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Representation of integral quantales by tolerances. (English) Zbl 1471.06009

Algebra Univers. 79, No. 1, Paper No. 5, 20 p. (2018).

A quantale is a triple $Q = (Q, \bigvee, \otimes)$, where $(Q, \bigvee)$ is a $\bigvee$-semilattice (partially ordered set having arbitrary joins), and $(Q, \otimes)$ is a semigroup such that the operation $\otimes$ distributes over arbitrary joins from both sides, namely, $a \otimes (\bigvee S) = \bigvee_{s\in S}(a \otimes s)$ and $(\bigvee S) \otimes a = \bigvee_{s\in S}(s \otimes a)$ for every element $a \in Q$ and every subset $S \subseteq Q$ [*K. I. Rosenthal*, Quantales and their applications. Harlow: Longman Scientific & Technical; New York: John Wiley & Sons, Inc. (1990; Zbl 0703.06007)]. A quantale Q is said to be unital provided that its underlying semigroup $(Q, \otimes)$ is a monoid, namely, has a unit element e. A unital quantale Q is integral provided that its unit is the top element $\top_Q$ of the $\bigvee$-semilattice $(Q, \bigvee)$. For example, for every unital quantale Q, the set $A = \{a \in Q \,|\, a \leqslant e\}$ is a subquantale of Q, which is an integral quantale (the integral part of Q).

There exist different representation theorems for quantales. For example, [*S. Valentini*, Math. Log. Q. 40, No. 2, 182–190 (1994; Zbl 0816.06018)] showed that every quantale Q is isomorphic to the quantale of the so-called ordered relations on Q. Taking on this result, the present paper provides a "more transparent" (as claimed by the authors) description of ordered relations and also shows how the reflexive ordered relations are connected to tolerances (reflexive and symmetric binary relations, compatible with the respective algebraic structure on the underlying set). Moreover, the authors prove that every integral quantale Q has a natural embedding into the quantale of complete tolerances on the underlying lattice of Q.

The paper additionally shows that the underlying lattice of any finite integral quantale Q is dually pseudocomplemented and distributive in $\top_Q$, where a lattice L is said to be pseudocomplemented provided that for every $a \in L$, there exists an element $a^* \in L$ (the pseudocomplement of a) such that for every $b \in L$, it follows that $b \wedge a = \bot_L$ if and only if $b \leqslant a^*$.

Lastly, the authors prove that certain relations on different algebraic structures naturally form a quantale, e.g., the set of all compatible reflexive binary relations on every finite majority algebra makes an integral quantale (we recall that a ternary term m of an algebra A is a majority term provided that the following identities hold on A: $m(x, x, y) = m(x, y, x) = m(y, x, x) = x$; for example, every algebra with a lattice reduct admits such a majority term; an algebra A admitting a majority term is called a majority algebra).

The paper is extremely well written, provides the most essential parts of its required preliminaries, and will be of interest to all those researchers who study the theory of quantales and especially their representation theorems.

Reviewer: Sergejs Solovjovs (Praha)

MSC:

Cited in **3** Documents

06F07 Quantales
06A15 Galois correspondences, closure operators (in relation to ordered sets)
06B15 Representation theory of lattices
06B23 Complete lattices, completions
06D22 Frames, locales

Keywords:

compatible relation; complete lattice; frame; join endomorphism; lattice ordered semiring; left translation; majority algebra; majority term; ordered relation; pseudocomplemented lattice; quantale; relatively pseudocomplemented lattice; residuated lattice; residuated pair; tolerance relation

Full Text: DOI

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