Moshier, M. A.; Pultr, A.; Suarez, A. L.
Exact and strongly exact filters. (English) Zbl 1461.18010
Appl. Categ. Struct. 28, No. 6, 907-920 (2020).

There exists a well-know notion of *frame*, i.e., a complete lattice, in which finite meets distribute over arbitrary joins [*P. T. Johnstone*, Stone spaces. Cambridge: Cambridge University Press (1982; Zbl 0499.54001)]. The paper considers the notion of (strongly) exact meet and its related concept of (strongly) exact filter in frames. A meet $\bigwedge A$ in a frame L is said to be *exact* provided that it distributes over joins, i.e., $(\bigwedge A) \vee b = \bigwedge_{a \in A}(a \vee b)$ for every $b \in L$. Moreover, $\bigwedge A$ is said to be *strongly exact* provided that it is preserved by every frame homomorphism h, i.e., $h(\bigwedge A) = \bigwedge_{a \in A} h(a)$ (every frame homomorphism preserves finite meets and arbitrary joins). The following implications for meets are valid in frames: "finite" ⇒ "strongly exact" ⇒ "exact". The classical notion of *filter* in a frame as an up-set (a subset A of a frame L such that for every $a \in A$ and every $b \in L$, $a \leqslant b$ implies $b \in A$) closed under finite meets can thus be easily extended to *(strongly) exact filters*, namely, as up-sets closed under (strongly) exact meets.

The paper provides a characterization of the sets of exact and strongly exact filters of a frame L, denoted $\mathrm{Filt}_{\mathrm{E}}(L)$ and $\mathrm{Filt}_{\mathrm{sE}}(L)$, respectively. One of the main characterization tools makes the notion of frame sublocale. A subset S of a frame L is a *sublocale* provided that it fulfills the following two properties: (1) if $M \subseteq S$, then $\bigwedge M \in S$; (2) if $s \in S$ and $a \in L$, then $a \to s \in S$, where the operation $\cdot \to \cdot$ is defined by $a \wedge b \leqslant c$ iff $a \leqslant b \to c$ for every $a, b, c, \in L$.

The authors show, in particular, that $\mathrm{Filt}_{\mathrm{sE}}(L)$ is naturally isomorphic to the system of the fitted sublocales of L (a sublocale is *fitted* provided that it is an intersection of *open* sublocales, namely, sublocales of the form $\{a \to b \,|\, b \in L\}$ for some $a \in L$), which addresses the question on representation of this system by filters considered in [*R. N. Ball* et al., Appl. Categ. Struct. 28, No. 4, 655–667 (2020; Zbl 1444.18018)]. The authors additionally show that the frame of exact filters $\mathrm{Filt}_{\mathrm{E}}(L)$ is a sublocale of the frame of strongly exact filters $\mathrm{Filt}_{\mathrm{sE}}(L)$.

The paper is well written and easy to read. It gives most of its required preliminaries, and will be of interest to the researchers studying point-free topology.

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MSC:

18F70	Frames and locales, pointfree topology, Stone duality
06A15	Galois correspondences, closure operators (in relation to ordered sets)
06D20	Heyting algebras (lattice-theoretic aspects)
06D22	Frames, locales

Keywords:

congruence; frame; Galois adjoint maps; Heyting structure; locale; localic map; nucleus; sober space; (strongly) exact filter; (strongly) exact meet; sublocale

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