Brazas, Jeremy; Fischer, Hanspeter
Test map characterizations of local properties of fundamental groups. (English)
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The classical theory of covering maps $p : (Y, y_0) \to (X, x_0)$ of basepointed spaces requests that X be path-connected, locally path-connected, and semilocally simply connected. Questions arise when only path-connectedness and local path-connectedness (generally speaking local path-connectedness is not necessarily stipulated for spaces under study herein) among these three are in play, and there are significant types of examples that fall into this classification. Numerous studies of such classes of spaces have been made; the authors provide many historical references to this in their introduction.

On page 38 is a list of six properties that a space might or might not have:

[1] Homotopically Hausdorff and its relative version,

[2] (transfinite products) Every homomorphism $f_\# : \pi_1(\mathbb{H}, b_0) \to \pi_1(X, x_0)$ induced by a map $f : \mathbb{H} \to X$ on the Hawaiian earring is uniquely determined by its values $f_\#([l_n])$ on the initial loops $[l_n]$,

[3] Existence of generalized universal and intermediate coverings,

[4] Homotopically path Hausdorff and its relative version,

[5] (1-UV_0) For every $x \in X$ and every neighborhood U of x there is an open set V in X with $x \in V \subseteq U$ and such that for every map $f : D^2 \to X$ from the unit disk with $f(\partial D^2) \subseteq V$, there is a map $g : D^2 \to U$ with $f : \partial D^2 = f|\partial D^2$,

[6] (π_1-shape injectivity) The canonical homomorphism $\pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$ to the first shape homotopy group is injective.

A diagram comparing properties (1), (3), (4), and (6) is provided in [*H. Fischer* et al., Topology Appl. 158, No. 3, 397–408 (2011; Zbl 1219.54028)].

The primary purpose of this paper is to provide a unified approach to comparing such properties as (1)–(6). We quote: "We are particularly motivated by the fact that even when X fails to admit a traditional universal covering, it is often the case that X admits a generalized universal covering [...] which acts in many ways as a suitable replacement. A generalized covering map is characterized only by its lifting properties and need not be a local homeomorphism."

A "diagram (on page 40) may serve as a reference for many of the results and definitions in this paper. It connects the relevant properties of a path-connected metric space X and the closure properties of a subgroup $H \leq \pi_1(X, x_0)$ [...] Equipped with this chart, we identify new types of subgroups that correspond to intermediate generalized coverings (Theorem 5.4 and Corollaries 7.14, 7.15) and shed more light on the relative position of the commutator subgroup of $\pi_1(\mathbb{H}, b_0)$ (Example 3.10). We also extend the existence of generalized universal coverings for Peano continua with residually n-slender fundamental group to all metric spaces (Corollary 6.5).

Property (5) is not an invariant of homotopy type, but is an important property held by one-dimensional and planar spaces and is known to imply the homotopically Hausdorff property for metric spaces. We improve the latter result by showing that every metric space with the 1-UV_0 property admits a generalized universal covering space (Theorem 6.9)."

Reviewer: Leonard R. Rubin (Norman)

MSC:

55Q52 Homotopy groups of special spaces
57M12 Low-dimensional topology of special (e.g., branched) coverings
06A15 Galois correspondences, closure operators (in relation to ordered sets)
54D05 Connected and locally connected spaces (general aspects)
55Q07 Shape groups
57M05 Fundamental group, presentations, free differential calculus

Keywords:

generalized covering space; homotopically Hausdorff; homotopically path Hausdorff; subgroup lattice of fundamental group; closure operator

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