El-Saady, Kamal; Sharqawy, Saad; Temraz, Ayat A.

Fuzzy quantic nuclei and conuclei with applications to fuzzy semi-quantales and (L, M)-quasi-fuzzy topologies. (English) Zbl 1454.06015

J. Egypt. Math. Soc. 27, Paper No. 28, 17 p. (2019).

There exists a convenient description of quotient quantales (resp. subquantales) with the help of quantic nuclei (resp. conuclei) [*K. I. Rosenthal*, Quantales and their applications. Harlow: Longman Scientific & Technical; New York: John Wiley & Sons, Inc. (1990; Zbl 0703.06007)]. Given a quantale Q, a *quantic nucleus* on Q is a map $j : Q \to Q$, which is a closure operator (an order-preserving map with $1_Q \leqslant j$ and $j \circ j = j$) such that $j(a) \otimes j(b) \leqslant j(a \otimes b)$ for every $a, b \in Q$, where $\otimes$ stands for the quantale operation on Q. A *quantic conucleus* changes property $1_Q \leqslant j$ to $j \leqslant 1_Q$ (relies on a coclosure operator).

Looking for the basic structure of lattice-valued topology, *S. E. Rodabaugh* [Int. J. Math. Math. Sci. 2007, Article ID 43645, 71 p. (2007; Zbl 1145.54004)] introduced the concept of *semi-quantale* as a partially ordered set having arbitrary joins and equipped with a binary operation $\otimes$ (with no additional requirement). Soon enough *M. Demirci* ["Fuzzy semi-quantales, (L, M) quasi-fuzzy topological spaces and their duality", in: Proceedings of the 7th international joint conference on computational intelligence (IJCCI), Lisbon, 2015. Piscataway, NJ: IEEE Press. 105–111 (2015)] provided a fuzzy version of semi-quantales in the sense of fuzzy groups of *A. Rosenfeld* [J. Math. Anal. Appl. 35, 512–517 (1971; Zbl 0194.05501)]. The present paper takes up the concept of Demirci [loc. cit.] and introduces its respective notion of fuzzy quantic (co)nucleus. The authors then show several ways of constructing fuzzy quantic (co)nuclei, relate them to (L, M)-fuzzy topology of *T. Kubiak* and *A. Šostak* [Quaest. Math. 20, No. 3, 423–429 (1997; Zbl 0890.54005)] (following the idea of Demirci [loc. cit.] that (L, M)-fuzzy topologies are actually fuzzy semi-quantales), and also to ideals of quantales of *S. Wang* and *B. Zhao* [J. Shaanxi Norm. Univ., Nat. Sci. Ed. 31, No. 4, 7–10 (2003; Zbl 1045.06007)], fuzzifying the latter concept (following the ideas of Rosenfeld again [loc. cit.]) to suit fuzzy semi-quantales.

While the paper is well written (the amount of typos is at the minimum), provides all of its required preliminaries, and could be of interest to the community of fuzzy algebraists, its mathematical content is a bit discouraging. First, the authors devote an entire section to introduce the well-known notion of product of quantales (including lengthy superfluous proofs). Second, not all proofs provided in the paper look correct (thus, some of the results seem doubtful, e.g., Proposition 4 on page 7, Lemma 3 on page 8, Proposition 8 on page 10, Proposition 11 on page 12, Proposition 15 on page 15, Lemmas 4, 5 on page 16). Third, the authors seem to be often in trouble with the notation of residuation operations in quantales. Recall that given a quantale $(Q, \otimes)$, every $a \in Q$ induces join-preserving maps $a \otimes -$ and $- \otimes a$, which thus have the respective upper adjoint maps $a \searrow -$ and $- \swarrow a$ (in the notation of the authors). These adjunctions then imply that for every $a, b, c \in Q$, $a \otimes b \leqslant c$ iff $b \leqslant a \searrow c$ iff $a \leqslant c \swarrow b$. However, Formula (1) on page 3 of the paper strangely gives "$a \otimes b \leqslant c \Leftrightarrow \ a \leqslant b \searrow c \Leftrightarrow \ b \leqslant c \swarrow a$". Fourth, it is not clear why the authors consider ideals of quantales, which are closed under *finite* joins, while quantales themselves rely on *infinite* joins, and, moreover, the ideals closed under infinite joins were used by Rosenthal [loc. cit.] himself (see the above citation). Fifth, almost at the very end of page 2, the authors state that "**CoQuant** is the full subcategory of **SQuant**, which has as objects all coquantales and as morphisms, all maps that preserve the tensor product and arbitrary meets." A full subcategory, however, singles out some objects and takes *all* morphisms between them (and this is the reason for calling it full).

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MSC:

06F07 Quantales
06A15 Galois correspondences, closure operators (in relation to ordered sets)
54A40 Fuzzy topology
06B30 Topological lattices

Keywords:

closure operator; fuzzy topology; ideal of quantale; interior operator; quantale; quantic nucleus

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