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Kostant's partition function and magic multiplex juggling sequences. (English)

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Summary: Kostant's partition function is a vector partition function that counts the number of ways one can express a weight of a Lie algebra \mathfrak{g} as a nonnegative integral linear combination of the positive roots of \mathfrak{g} . Multiplex juggling sequences are generalizations of juggling sequences that specify an initial and terminal configuration of balls and allow for multiple balls at any particular discrete height. Magic multiplex juggling sequences generalize further to include magic balls, which cancel with standard balls when they meet at the same height. In this paper, we establish a combinatorial equivalence between positive roots of a Lie algebra and throws during a juggling sequence. This provides a juggling framework to calculate Kostant's partition functions, and a partition function framework to compute the number of juggling sequences. From this equivalence we provide a broad range of consequences and applications connecting this work to polytopes, posets, positroids, and weight multiplicities.

MSC:

- 05E10 Combinatorial aspects of representation theory
- 05A15 Exact enumeration problems, generating functions
- 05A18 Partitions of sets
- 17B22 Root systems
- 17B99 Lie algebras and Lie superalgebras

Keywords:

Kostant's partition function; multiplex juggling sequence; magic juggling sequence; juggling; juggling polytope; juggling poset

Software:

lie_algebras

Full Text: DOI

References:

- [1] Armstrong, D.; Garsia, A.; Haglund, J.; Rhoades, B.; Sagan, B., Combinatorics of Tesler matrices in the theory of parking functions and diagonal harmonics, J. Comb., 3, 3, 451-494 (2012) · Zbl 1291.05203 · doi:10.4310/JOC.2012.v3.n3.a7
- [2] Ayer, A., Bouttier, J., Corteel, S., Nunzi, F., Multivariate juggling probabilities, Proceedings of the 25th International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms,(2014), pp. 1-12 · Zbl 1331.60021
- [3] Baldoni, W.; Vergne, M., Kostant partitions functions and flow polytopes, Transform. Groups, 13, 3-4, 447-469 (2008) · Zbl 1200.52008 · doi:10.1007/s00031-008-9019-8
- [4] Baldoni, M. W., Beck, M., Cochet, C., Vergne, M., Volume computation for polytopes and partition functions for classical root systems, Discrete Comput. Geom., 35, (2006), no. 4, 551-595 Code available at www.math.polytechnique.fr/cmat/vergne/. Accessed 9 Jan 2020 · Zbl 1105.52001
- [5] Beck, M., Robins, S., Computing the continuous discretely. Undergraduate texts in mathematics (2nd edition), Springer, New York, (2015). 10.1007/978-1-4939-2969-6 · Zbl 1339.52002
- [6] Benedetti, C.; González D'León, RS; Hanusa, CRH; Harris, P.; Khare, A.; Morales, A.; Yip, M., A combinatorial model for computing volumes of flow polytopes, Trans. Amer. Math. Soc., 372, 3369-3404 (2019) · Zbl 1420.05011 · doi:10.1090/tran/7743
- [7] Buhler, J.; Eisenbud, D.; Graham, R.; Wright, C., Juggling drops and descents, Amer. Math. Monthly, 101, 6, 507-519 (1994) · Zbl 0814.05002 · doi:10.2307/2975316
- [8] Butler, S., Private email conversation, June 2018
- [9] Butler, S., Choi, J., Kim, K., Seo, K., Enumerating multiplex juggling patterns, J. Integer Seq., 22, (2019), no.1, Art. 19.1.7, 21 · Zbl 1407.05011

- [10] Butler, S.; Graham, R., Enumerating (Multiplex) Juggling Sequences, *Ann. Comb.*, 13, 4, 413-424 (2010) · [Zbl 1231.05009](#) · [doi:10.1007/s00026-009-0040-y](#)
- [11] Chan, C.; Robbins, D.; Yuen, DS, On the volume of a certain polytope, *Exp. Math.*, 9, 1, 91-99 (2000) · [Zbl 0960.05004](#) · [doi:10.1080/10586458.2000.10504639](#)
- [12] Chang, K.; Harris, PE; Insko, E., Kostant's weight multiplicity formula and the Fibonacci and Lucas numbers, *J. Comb.*, 11, 1, 141-167 (2020) · [Zbl 1428.17004](#) · [doi:10.4310/JOC.2020.v11.n1.a7](#)
- [13] Chung, F.; Graham, R., Primitive juggling sequences, *Amer. Math. Monthly.*, 115, 3, 185-194 (2008) · [Zbl 1170.05006](#) · [doi:10.1080/00029890.2008.11920516](#)
- [14] Corteel, S.; Kim, JS; Mészáros, K., Flow polytopes with Catalan volumes, *C. R. Math. Acad. Sci. Paris*, 355, 3, 248-259 (2017) · [Zbl 1358.05129](#) · [doi:10.1016/j.crma.2017.01.007](#)
- [15] Corteel, S., Kim, J. S., Mészáros, K., Volumes of Generalized Chan-Robbins-Yuen Polytopes, *Discrete Comput. Geom.*, 2019, 10.1007/s00454-019-00066-1
- [16] Ehrenborg, R., Determinants involving (q) -Stirling numbers, *Adv. in Appl. Math.*, 31, (2003), no. 4, 630-642, 10.1016/S0196-8858(03)00029-0 · [Zbl 1071.05011](#)
- [17] Ehrenborg, R. , Readdy, M., Juggling and applications to (q) -analogues, *Proceedings of the 6th Conference on Formal Power Series and Algebraic Combinatorics* (New Brunswick, NJ, 1994), 1996, pp. 107-125, 10.1016/S0012-365X(96)83010-X · [Zbl 0859.05010](#)
- [18] Engström, A., Leskelä, L., Varpanen, H., Geometric juggling with (q) -analogues, *Discrete Math.*, 338, (2015), no. 7, 1067-1074, 10.1016/j.disc.2015.02.004 · [Zbl 1309.05028](#)
- [19] Goodman, R.; Wallach, NR, *Symmetry, Representations and Invariants* (2009), New York: Springer, New York · [Zbl 1173.22001](#)
- [20] Haglund, J., A polynomial expression for the Hilbert series of the quotient ring of diagonal coinvariants, *Adv. Math.*, 227, 5, 2092-2106 (2011) · [Zbl 1258.13020](#) · [doi:10.1016/j.aim.2011.04.013](#)
- [21] Harris, P. E., On the adjoint representation of (\mathfrak{sl}_n) and the Fibonacci numbers, *C. R. Math. Acad. Sci. Paris*, 349, (2011), 935-937, 10.1016/j.crma.2011.08.017 · [Zbl 1273.17010](#)
- [22] Harris, PE, *Combinatorial problems related to Kostant's weight multiplicity formula* (2012), Milwaukee: University of Wisconsin, Milwaukee
- [23] Harris, P. E., Insko, E., Omar, M., The (q) -analog of Kostant's partition function and the highest root of the simple Lie algebras, *Australas. J. Combin.*, 71, (2018), no. I, 68-91 · [Zbl 1459.17021](#)
- [24] Harris, P. E., Insko, E., Simpson, A., Computing weight (q) -multiplicities for the representations of the simple Lie algebras, *A. AAEECC*, 2017, 10.1007/s00200-017-0346-7 · [Zbl 1436.17011](#)
- [25] Harris, PE; Insko, E.; Williams, LK, The adjoint representation of a classical Lie algebra and the support of Kostant's weight multiplicity formula, *J. Comb.*, 7, 1, 75-116 (2016) · [Zbl 1330.05164](#)
- [26] Harris, P. E., Lauber, E., Weight (q) -multiplicities for representations of $(\mathfrak{sp}_4(\mathbb{C}))$, *Zh. Sib. Fed. Univ. Math. Phys.*, 10, (2017), no. 4, 494-502, 10.17516/1997-1397-2017-10-4-494-502
- [27] Harris, PE; Lescinsky, H.; Mabie, G., Lattice patterns for the support of Kostant's weight multiplicity formula on $(\mathfrak{sl}_3(\mathbb{C}))$, *Minnesota Journal of Undergraduate Mathematics.*, 4, 1 (2018)
- [28] P. E. Harris, M. Loving, J. Ramirez, J. Rennie, G. Rojas Kirby, E. Torres Davila, F. O. Ulysse, Visualizing the support of Kostant's weight multiplicity formula for the rank two Lie algebras, (2019), arXiv preprint arXiv:1908.08405.pdf
- [29] Hille, L., Quivers, cones and polytopes, *Linear Algebra Appl.*, 365, 215-237 (2003) · [Zbl 1034.52011](#) · [doi:10.1016/S0024-3795\(02\)00406-8](#)
- [30] Jang, J., Kim, J. S., Volumes of flow polytopes related to caracol graphs, 2019, arXiv preprint arXiv:1911.10703
- [31] Kapoor, K., Mészáros, K., Setiabrata, L., Counting integer points of flow polytopes, 2019, arXiv preprint arXiv:1906.05592,
- [32] Knutson, A.; Lam, T.; Speyer, D., Positroid varieties: juggling and geometry, *Compos. Math.*, 149, 1710-1752 (2013) · [Zbl 1330.14086](#) · [doi:10.1112/S0010437X13007240](#)
- [33] Kostant, B., A formula for the multiplicity of a weight, *Proc. Nat. Acad. Sci. U.S.A.*, 44, 588-589 (1958) · [Zbl 0081.02202](#) · [doi:10.1073/pnas.44.6.588](#)
- [34] Leskelä, L.; Varpanen, H., Juggler's exclusion process, *J. Appl. Probab.*, 49, 1, 266-279 (2012) · [Zbl 1242.60103](#) · [doi:10.1239/jap/1331216846](#)
- [35] Lidskiĭ, BV, The Kostant function of the system of roots (A_n) , *Funktsional. Anal. i Prilozhen.*, 18, 1, 76-77 (1984) · [Zbl 0542.22014](#) · [doi:10.1007/BF01076370](#)
- [36] Mészáros, K.; Morales, AH, Flow polytopes of signed graphs and the Kostant partition function, *Int. Math. Res. Not. IMRN*, 3, 830-871 (2015) · [Zbl 1307.05097](#) · [doi:10.1093/imrn/rnt212](#)
- [37] Mészáros, K.; Morales, AH; Striker, J., On Flow Polytopes, Order Polytopes, and Certain Faces of the Alternating Sign Matrix Polytope, *Discrete Comput. Geom.*, 62, 1, 128-163 (2019) · [Zbl 1414.05029](#) · [doi:10.1007/s00454-019-00073-2](#)
- [38] Liu, R. I., Mészáros, K., St. Dizier, A., Gelfand-Tsetlin polytopes: a story of flow and order polytopes, 2019, arXiv preprint arXiv:1903.08275 · [Zbl 1428.05319](#)
- [39] Mészáros, K., Product formulas for volumes of flow polytopes, *Proc. Amer. Math. Soc.*, 143, 3, 937-954 (2015) · [Zbl 1310.51024](#) · [doi:10.1090/S0002-9939-2014-12182-4](#)
- [40] Mészáros, K.; Morales, AH, Volumes and Ehrhart polynomials of flow polytopes, *Math. Z.*, 293, 3-4, 1369-1401 (2019) · [Zbl 1453.52012](#) · [doi:10.1007/s00209-019-02283-z](#)

- [41] K. Mészáros, A. H. Morales, B. Rhoades, The polytope of Tesler matrices, *Selecta Math. (N.S.)*, 23, (2017), no.1, 425-454, 10.1007/s00029-016-0241-2 · [Zbl 1355.05271](#)
- [42] Mészáros, K., Simpson, C., Wellner, Z., Flow polytopes of partitions, *Electron. J. Combin.*, 26, (2019), no. 1, Paper 1.47, 12 · [Zbl 1409.52013](#)
- [43] Mezzarobba, M.; Salvy, B., Effective bounds for P-recursive sequences, *J. Symbolic Comput.*, 45, 10, 1075-1096 (2010) · [Zbl 1201.65219](#) · [doi:10.1016/j.jsc.2010.06.024](#)
- [44] Polster, B., *The mathematics of juggling* (2003), New York: Springer-Verlag, New York · [Zbl 1116.00004](#)
- [45] Schmidt, JR; Bincer, AM, The Kostant partition function for simple Lie algebras, *Journal of Mathematical Physics*, 25, 2367-2373 (1984) · [Zbl 0551.17002](#) · [doi:10.1063/1.526457](#)
- [46] J. O'Neill, On the poset and asymptotics of Tesler matrices, *Electron. J. Combin.*, 25, (2018), no.2, Paper 2.4, 27
- [47] C. Shannon, *Scientific Aspects of Juggling*, (1980), Published in (Wiley 1993), 850864, Retrieved from <https://www.jonglage.net/theorie/notation/avancee/refs/Claude%20Shannon%20-%20Scientific%20Aspects%20of%20Juggling.pdf>
- [48] Simpson, A., *Kostant's Partition Function and Multiplex Juggling Sequences*, Senior Thesis, Department of Mathematics and Statistics, Williams College, (2019), Available at <https://unbound.williams.edu/theses/islandora/object/studenttheses%3A1382/> accessed 12-June-2019
- [49] Stadler, JD, Juggling and vector compositions, *Discrete Math.*, 258, 1-3, 179-191 (2002) · [Zbl 1009.05011](#) · [doi:10.1016/S0012-365X\(02\)00269-8](#)
- [50] Stanley, R. P., *Enumerative combinatorics. Volume 1, Second*, Cambridge Studies in Advanced Mathematics, vol.49, Cambridge University Press, Cambridge, 2012
- [51] Stanley, R. P., *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol.62, Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin, 10.1017/CBO9780511609589
- [52] H. Varpanen, *A short publication history of juggling math*, (2012), Retrieved from <https://www.semanticscholar.org/paper/A-short-publication-history-of-juggling-math-Varpanen/887ab6516746828315bf534e9443e66fc531ce2e>
- [53] Yip, M., *A Fuss-Catalan variation of the caracol flow polytope*, (2019), arXiv preprint arXiv:1910.10060
- [54] Zeilberger, D., Proof of a conjecture of Chan, Robbins, and Yuen, *Electron. Trans. Numer. Anal.*, 9, (1999), 147-148 · [Zbl 0941.05006](#)

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