Gouveia, Maria João; Santocanale, Luigi
The continuous weak order. (English) Zbl 1446.18004
J. Pure Appl. Algebra 225, No. 2, Article ID 106472, 37 p. (2021).

The paper studies a generalization of the weak Bruhat order on the set of permutations of a set with n elements, also known as permutohedron (see, e.g., [*N. Caspard* et al., in: Lattice theory: special topics and applications. Volume 2. Basel: Birkhäuser/Springer. 215–286 (2016; Zbl 1401.06003)] for more details).

One of the natural generalizations of permutohedra are multinomial lattices (introduced in [*M. K. Bennett* and *G. Birkhoff*, Algebra Univers. 32, No. 1, 115–144 (1994; Zbl 0810.06006)] as part of an order-theoretic investigation of rewrite systems associated with common algebraic laws). Elements of a multinomial lattice are multipermutations, i.e., words on a totally ordered finite alphabet $\Sigma = \{a, b, c, \dots\}$ with a fixed number of occurrences of each letter. The weak order on multipermutations is the reflexive and transitive closure of the binary relation $\prec$ defined by $wabu \prec wbau$ for $a, b \in \Sigma$ such that $a < b$. If each letter of the alphabet has exactly one occurrence, then these words are permutations, and the ordering is the weak Bruhat ordering.

Multipermutations can be given a geometrical interpretation as discrete increasing paths in some Euclidean cube of dimension $d = \mathrm{card}(\Sigma)$, and the weak order can be considered as a way of making these paths into a lattice structure. When $\mathrm{card}(\Sigma) = 2$, the connection with geometry is well-established, i.e., these lattices are also known as lattices of lattice paths [*L. Ferrari* and *R. Pinzani*, J. Stat. Plann. Inference 135, No. 1, 77–92 (2005; Zbl 1082.06006)]. The present paper answers the question on whether the weak order can be extended from discrete paths to continuous increasing paths.

The authors consider the quantale $Q_\vee(\mathbb{I})$ of join-preserving maps from the unit interval $\mathbb{I} = [0, 1]$ to itself (with the standard structure of map composition as quantale tensor and the identity map as quantale unit; see, e.g., [*D. Kruml* and *J. Paseka*, Handb. Algebra 5, 323–362 (2008; Zbl 1219.06016)] for more details on quantales). With the notation $[d]_2 = \{(i, j) \mid 1 \leqslant i \leqslant j \leqslant d\}$ for $d \geqslant 2$, they show that certain elements (called clopen (closed and open) tuples and denoted $\mathsf{L}_d(Q_\vee(\mathbb{I}))$) of the product quantale $Q_\vee(\mathbb{I})^{[d]_2}$ are in bijective correspondence with images of monotone increasing continuous maps $p : \mathbb{I} \to \mathbb{I}^d$ such that $p(0) = \vec{0}$ and $p(1) = \vec{1}$ (called paths). The obtained lattice-theoretic structure on paths is then said to be the continuous weak order in dimension d. The authors additionally show that the construction $\mathsf{L}_d(-)$ gives a limit-preserving functor from a certain category of lattice-ordered semigroups to the category of lattices.

The paper also has a section devoted to the algebraic structure of the lattices of the form $\mathsf{L}_d(Q_\vee(\mathbb{I}))$, namely, it characterizes their join-irreducible elements and also shows that these lattices have neither completely join-irreducible elements (an element a of a complete lattice L is called *completely join-irreducible* provided that for every subset $S \subseteq L$, $a = \bigvee S$ implies $a \in S$) nor compact elements (an element $c \in L$ is said to be *compact* provided that for every directed subset $D \subseteq L$, $c \leqslant \bigvee D$ implies $c \leqslant d$ for some $d \in D$).

The paper is well written and contains most of its required preliminaries. Despite its rather technical nature (and a number of lengthy proofs), the paper will be of interest to all the researchers doing lattice theory.

Reviewer: Sergejs Solovjovs (Praha)

MSC:

18B35	Preorders, orders, domains and lattices (viewed as categories)
05A05	Permutations, words, matrices
06A15	Galois correspondences, closure operators (in relation to ordered sets)
06B23	Complete lattices, completions
06F07	Quantales
18F75	Quantales
52B99	Polytopes and polyhedra

Keywords:

⋆-autonomous quantale; compact element; continuous weak order; Dedekind-MacNeille completion; join-continuous map; join-irreducible element; join-prime element; lattice-ordered semigroup; MIX rule; multinomial lattice; path; permutohedron; residuated lattice; skew metric; weak Bruhat order

Full Text: DOI arXiv

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