Shir Ali Nasab, Amir Reza; Hosseini, Seyed Naser
Connectedness in a category. (English) Zbl 1446.18002
Bull. Iran. Math. Soc. 46, No. 4, 1195-1210 (2020).

There exists a convenient categorical generalization of the concept of closure operator in a topological space (see, for example, [*D. Dikranjan* and *W. Tholen*, Categorical structure of closure operators. With applications to topology, algebra and discrete mathematics. Dordrecht: Kluwer Academic Publishers (1995; Zbl 0853.18002)] for a thorough description of the topic). More precisely, a *closure operator* C in a category $\mathbf{X}$ with respect to a class $\mathcal{M}$ of $\mathbf{X}$-subobjects is given by a family $C = (c_X)_{X\in Ob(\mathbf{X})}$ of maps $c_X : \mathcal{M}/X \to \mathcal{M}/X$ (where $\mathcal{M}/X$ is a partially ordered class of isomorphism classes of monomorphisms in $\mathcal{M}$ with codomain X) such that for every $\mathbf{X}$-object X, the following three conditions hold:

(1) (Extension) $m \leqslant c_X(m)$ for every $m \in \mathcal{M}/X$;

(2) (Monotonicity) if $m \leqslant m'$, then $c_X(m) \leqslant c_X(m')$ for every m, $m' \in \mathcal{M}/X$; and, finally,

(3) (Continuity) $c_X(f^{-1}(m)) \leqslant f^{-1}(c_Y(m))$ for every $f : X \to Y$ and every $m \in \mathcal{M}/Y$ (in which $f^{-1}(-) : \mathcal{M}/Y \to \mathcal{M}/X$ is the *inverse image* map given on an element $m \in \mathcal{M}/Y$ by a pullback along the morphism $f : X \to Y$ in question).

Given a closure operator C in a category $\mathbf{X}$, an element $m \in \mathcal{M}/X$ is called C-*closed* (resp. C-*dense*) provided that $c_X(m) = m$ (resp. $c_X(m) = 1_X$, where 1_X stands for the identity morphism on X).

The above notion of categorical closure operator allows one to extend the concept of connected topological space to an object of a suitable category (see, e.g., [*M. M. Clementino* and *W. Tholen*, Topology Appl. 75, No. 2, 143–181 (1997; Zbl 0906.18003); Appl. Categ. Struct. 9, No. 6, 539–556 (2001; Zbl 0993.18004); *J. Šlapal*, Appl. Categ. Struct. 17, No. 6, 603–612 (2009; Zbl 1184.54014)]). The present paper follows suit and presents yet another approach to connectedness via closure operators. In particular, it defines a suitable class $\mathcal{M}$ of monomorphisms, which is not necessarily a part of a factorization structure on a category $\mathbf{X}$. More precisely, the paper calls a class $\mathcal{M}$ of monomorphisms in $\mathbf{X}$ a *domain* provided that:

(1) $\mathcal{M}$ contains all the identities;

(2) $\mathcal{M}$ is stable under pullbacks, i.e., for every $m \in \mathcal{M}/X$ and every $f : Y \to X$ in $\mathbf{X}$, a pullback $f^{-1}(m)$ of m along f exists and lies in $\mathcal{M}/Y$;

(3) for every $\mathbf{X}$-object X, $\mathcal{M}/X$ is closed under binary meets (where the meet of a, b is the diagonal of a pullback of a along b);

(4) for every $\mathbf{X}$-object X, $\mathcal{M}/X$ has the smallest element.

The definition of connectedness given in the paper is based in the idea that a topological space X is connected provided that there do not exist two non-empty closed subsets S_1 and S_2 of X with the property that $S_1 \cap S_2 = \varnothing$ and $X = S_1 \cup S_2$ (or equivalently $X\backslash(S_1 \cup S_2) = \varnothing$). The authors then illustrate their notion with several examples and generalize two well-known topological results: on connectedness of union and product of connected topological spaces.

The paper is well written, provides most of its required preliminaries, and could be of interest to all those researchers, who study categorical topology.

Reviewer: Sergejs Solovjovs (Praha) (MR4125953)

MSC:

18A20	Epimorphisms, monomorphisms, special classes of morphisms, null morphisms
06A15	Galois correspondences, closure operators (in relation to ordered sets)
18B99	Special categories
18C40	Structured objects in a category (group objects, etc.)
18F60	Categories of topological spaces and continuous mappings
54D05	Connected and locally connected spaces (general aspects)

Keywords:

closure operator; connected topological space; domain; factorization structure; fine epimorphism; monomorphism; preorder; product of objects; pseudo-complement; pullback; quasi-complement; strict initial object

Full Text: DOI

References:

[1] Adamek, J.; Herrlich, H.; Strecker, GE, Abstract and Concrete Categories (1990), New York: Wiley, New York

[2] Castellini, G., Connectedness with respect to a closure operator, Appl. Categ. Struct., 9, 285-302 (2001) · Zbl 0981.18009 · doi:10.1023/A:1011247621101

[3] Castellini, G., Categorical Closure Operators (2003), Boston: Birkhäuser, Boston · Zbl 1045.18001

[4] Castellini, G.; HajekClosure, D., Operators and connectedness, Topol. Appl., 55, 29-45 (1994) · Zbl 0791.54001 · doi:10.1016/0166-8641(94)90063-9

[5] Castellini, G.; Holgate, D., A link between two connectedness notions, Appl. Categ. Struct., 11, 473-486 (2003) · Zbl 1039.18001 · doi:10.1023/A:1025732820692

[6] Clementino, MM, On connectedness via closure operator, Appl. Categ. Struct, 9, 539-556 (2001) · Zbl 0993.18004 · doi:10.1023/A:1012512306420

[7] Clementino, MM; Tholen, W., Separation versus connectedness, Topol. Appl., 75, 143-181 (1997) · Zbl 0906.18003 · doi:10.1016/S0166-8641(96)00087-9

[8] Dikranjan, D.; Tholen, W., Categorical Structure of Closure Operators (1995), Amsterdam: Kluwer Academic Publishers, Amsterdam · Zbl 0853.18002

[9] Enochs, EE; Jenda, OMG, Relative Homological Algebra (2000), Berlin: Walter de Gruyter, Berlin

[10] Maclane, S.; Moerdijk, I., Sheaves in Geometry and Logic, A First Introduction to Topos Theory (1992), Berlin: Springer, Berlin

[11] Mousavi, S. Sh; Hosseini, S. N., Quasi right factorization structures as presheaves, Appl. Categ. Struct., 19, 741-756 (2011) · Zbl 1237.18002 · doi:10.1007/s10485-010-9242-z

[12] Preuss, G., Relative connectedness and disconnectedness in topological categories, Quaest. Math., 2, 297-306 (1977)· doi:10.1080/16073606.1977.963

[13] Slapal, J., Another approach to connectedness with respect to a closure operator, Appl. Categ. Struct., 17, 603-612 (2009) · Zbl 1184.54014 · doi:10.1007/s10485-008-9163-2