Zhao, Bin; **Xia, Changchun**

Precoherent quantale completions of partially ordered semigroups. (English) Zbl 1444.06011

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There exists the concept of coherent locale, and it is well-known that a locale is coherent if and only if it is isomorphic to the locale of ideals of a distributive lattice (cf., e.g., [*P. T. Johnstone*, Stone spaces. Cambridge: Cambridge University Press (1982; Zbl 0499.54001)]). A non-commutative version of this result states that a quantale is coherent if and only if it is isomorphic to the quantale of all ideals of an m-semilattice (Proposition 4.2 on page 209 of [*J. Paseka*, Arch. Math., Brno 22, 203–210 (1986; Zbl 0612.06012)]; see also [*K. Keimel*, Acta Math. Acad. Sci. Hung. 23, 51–69 (1972; Zbl 0265.06016)]). The present paper studies those completions of partially ordered semigroups, which give precoherent quantales. Recall that a quantale $(Q,\otimes,\leqslant)$ is said to be *precoherent* provided that $(Q,\leqslant)$ is an algebraic lattice, and $K(Q)$ (the set of compact elements of Q) is closed under $\otimes$. A precoherent quantale Q is then called *coherent* provided that the top element $\top_Q$ of Q is compact (Definition 2.9 on page 620).

The paper first provides a representation theorem for coherent quantales, namely, shows that a quantale Q is coherent if and only if Q is isomorphic to the quantale of all distributive ideals of a partially ordered semigroup with a top element (Theorem 3.9 on page 623). Notice that in case of m-semilattices, distributive ideals are precisely ideals, which gives back the above-mentioned result for m-semilattices. The next result shows that precoherent quantale completions of a partially ordered semigroup S are in one-to-one correspondence with quantic quotients of the quantale $\mathcal{P}(S)$ (the powerset of S) with respect to algebraic consistent quantic nuclei on $\mathcal{P}(S)$ (Corollary 3.16 on page 625). Moreover, Theorem 3.24 on page 627 gives the smallest and the largest precoherent quantale completion of S (notice that the set of all algebraic consistent quantic nuclei on $\mathcal{P}(S)$ is a complete lattice and thus has the smallest and the largest element). Additionally, the authors provide the necessary and sufficient conditions on S for its Frink completion to be a coherent quantale (Corollary 3.33 on page 629). Lastly, motivated by the result that quantales are precisely the injective objects in a certain category of partially ordered semigroups (namely, by Theorem 4 on page 374 of [*X. Zhang* and *V. Laan*, Proc. Est. Acad. Sci. 63, No. 4, 372–378 (2014; Zbl 1332.06047)], which relies on the arguments of Theorem 4.1 on page 342 of [*J. Lambek* et al., Theory Appl. Categ. 26, 338–348 (2012; Zbl 1259.06019)]), the authors define a category of algebraic partially ordered semigroups $\mathbf{APoSgr}_{\leq}$ and a class of its morphism $\mathscr{E}_{\leq}$ such that $\mathscr{E}_{\leq}$-injective objects in $\mathbf{APoSgr}_{\leq}$ are precisely precoherent quantales (Theorem 4.4 on page 631). The authors, however, claim to be unable to find $\mathscr{E}_{\leq}$-injective hulls in $\mathbf{APoSgr}_{\leq}$ (Remark 4.7(1) on page 632).

The paper is sufficiently well written (with a number of typos, like, e.g., "pr*o*coherent" instead of "pr*e*coherent" on page 618), contains nearly all of its required preliminaries, and will be of interest to all the researchers, who work with the theory of ordered algebraic structures.

Reviewer: Sergejs Solovjovs (Praha)

MSC:

06F07	Quantales
06A15	Galois correspondences, closure operators (in relation to ordered sets)
06B10	Lattice ideals, congruence relations
06B23	Complete lattices, completions
06F05	Ordered semigroups and monoids
18B99	Special categories

Keywords:

algebraic lattice; closure operator; compact element of a poset; coherent quantale; completion of a partially ordered semigroup; distributive ideal; Frink ideal; injective object; partially ordered semigroup; quantale; quantic nucleus

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