The title of this work includes two terms, “contextuality”, and “noncommutative geometry”, both applied to quantum mechanics. The latter is quite understandable for the reader, I believe, whereas the former has many different meanings. In view of quantum mechanics, either the term “contextual” or “(non)contextual” has one particular meaning which is due to J. S. Bell [Rev. Mod. Phys. 38, 447–452 (1966; Zbl 0152.23605)], though, as R. B. Griffiths noted [“Quantum measurements and contextuality”, Preprint, arXiv:1902.05633], Bell did not use this term which was introduced later [A. Shimony, “Hidden-variable models of quantum mechanics (nocontextual and contextual)”, in: Compendium of quantum physics. Berlin: Springer. 287–291 (2009); A. Khrennikov, Contextual approach to quantum formalism. Dordrecht: Springer (2009; Zbl 1176.81001)].

Actually, ‘(non)contextuality’ links to the topic of quantum measurements that is another term in quantum foundations that has no “clear meaning” ([A. Acín et al., Commun. Math. Phys. 334, No. 2, 533-628 (2015; Zbl 1312.81010)]) in the following manner: Let $A, B,$ and $C$ be three quantum observables. Let us assume that $A$ commutes with $B$ and with $C$, whereas $B$ does not commute with $C$. Therefore, it is possible to simultaneously measure $A$ along with $B$, or $A$ along with $C$, but one cannot measure all three, $A, B,$ and $C$ at the same time (in a single experimental run). Bell then questioned of, first, whether a measurement of $A$ along with $B$ demands a different apparatus than a measurement of $A$ along with $C$, and second, would the value obtained for $A$ be the same when measured with $B$ as when measured with $C$, and concluded that if the answer is “yes”, quantum mechanics is noncontextual, and if the answer is “no”, or at least “no” in some cases, quantum mechanics is contextual in the sense that the measurement outcome for $A$ depends upon whether it is measured along with $B$, the $\{A, B\}$ context or together with $C$, the $\{A, C\}$ context.

In this relation, let us recall the famous foundational debates between N. Bohr, on the one side, and A. Einstein, B. Podolsky, and N. Rosen, on the other, that is well-known as the EPR Gedanken-experiment [A. Einstein et al., Phys. Rev., II. Ser. 47, 777–780 (1935; Zbl 0012.04201)] on whether the quantum state does provide ‘a complete description’ of a system (see my previous review in Zbmath ...). These debates led to the hidden variable models of quantum theory – i.e., the models in which quantum states are represented as probability distributions over a space of more fundamental ontic states that yield deterministic values for all observables. The authors of the present paper write then: “Motivated by a desire to hold onto realism, one may insist that a hidden variable model be noncontextual – that the values of the system’s observable properties be independent of the precise method of observation, and, in particular, of which other observables are measured simultaneously.” However, S. Kochen and E. P. Specker [J. Math. Mech. 17, 59–87 (1967; Zbl 0156.23302)] rules out hidden variable models of this kind, thus demonstrating that contextuality is a necessary feature of any theory that reproduces the highly-verified empirical predictions of quantum mechanics.

This is one line of the work under review that is reflected in its title. The other, that comes from the term ‘noncommutative geometry’, primarily motivates this work that aims, first, to study a candidate geometric notion of state space for quantum systems, second, to identify and to explore a connection with noncommutative geometry in order to build such geometric construction that necessarily accounts for contextuality as an obstacle towards a naively ontological quantum state space.

To achieve the aforementioned goal, the authors start with the spectral presheaf formulation of the Bell-Kochen-Specker theorem based on the following observations: (i) [J. Hamilton et al., Int. J. Theor. Phys. 39, No. 6, 1413–1436 (2000; Zbl 1055.81004); C. J. Isham and J. Butterfield, Int. J. Theor. Phys. 37, No. 11, 2669–2733 (1998; Zbl 0979.81018)] associate to a von Neumann algebra a presheaf of compact Hausdorff spaces, varying over contexts (commutative von Neumann subalgebras representing sets of jointly measurable observables); (ii) The Bell-Kochen-Specker theorem results in the expression as the
nonexistence of a global section of points (i.e. a global point in the generalized ‘space’), whereas Gleason’s theorem [A. M. Gleason, J. Math. Mech. 6, 885–893 (1957; Zbl 0078.28803)] can be expressed as a correspondence between quantum states and global sections of probability distributions (i.e., a global probability distribution on the generalized ‘space’), which, as they strongly believe, might play a notion of quantum state space that fundamentally incorporates contextuality. All this dictates the structure of the present work which consists of seven sections and three appendices (the latter are entitled as ‘A. Concrete Colimit Construction’, ‘B. Topological and $C^*$-Algebraic $K$-Theory’, and ‘C. Ideals of Operator Algebras’, correspondingly). Section 2 surveys the main background topics, such as state-observable dualities, quantum contextuality, the spectral presheaf, and noncommutative geometry. In Section 3, the authors introduce the technical necessary machinery for contravariantly functionally associating diagrams of topological spaces that describe quotient spaces of a ‘noncommutative space’ to noncommutative operator algebra. Section 4 entitled ‘Extensions of Topological Functors’ gives a generalization of limit and colimit functors acting on certain functor categories to those which act on categories of diagram, and defines the extension of a topological functor to a noncommutative algebraic one, given a semispectral functor which is described in Section 3. This construction is illustrated by presenting formulations of the generalized Bell-Kochen-Specker and Gleason’s theorems. The next Section 5 considers the extension of the topological $K$-theory functor. In Section 6 the authors formalize the idea of using extensions to directly obtain noncommutative analogues from basic topological concepts in order to establish the conjecture that extending the topological notion of closed subset leads to its algebraic generalization: closed two-sided ideal. Section 7 concludes and summarizes this work.

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**MSC:**

81P13 Contextuality in quantum theory  
81R60 Noncommutative geometry in quantum theory  
46L87 Noncommutative differential geometry  
81P15 Quantum measurement theory, state operations, state preparations  
81P16 Quantum state spaces, operational and probabilistic concepts  
81P40 Quantum coherence, entanglement, quantum correlations  
81P65 Quantum gates  
18F20 Presheaves and sheaves, stacks, descent conditions (category-theoretic aspects)  
54B40 Presheaves and sheaves in general topology  
46L05 General theory of $C^*$-algebras  
46L30 States of selfadjoint operator algebras

**Keywords:**  
$C^*$-algebra; quantum mechanics; observable; measurement theory; Gleason’s theorem; contextuality; noncommutative geometry; Bell-Kochen-Specker theorem; state-observable duality; spectral presheaf; compact Hausdorff topological space; $K$-theory

**Full Text:** DOI

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