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An order-theoretic perspective on categorial closure operators. (English) Zbl 1406.18001

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The central contribution of this paper is to show how categorical closure operations (cco's) are strictly connected with binary closure operators (bco's), the specific instance of cco's in posets. This fact reveals hidden links among many algebraic, topological, and categorical notions.

Intuitively, a cco is the categorical structure that abstracts over closure operators in topology, mapping a subspace to the minimal closed subspace which contains it. Conversely, a bco is an algebraic partial operation $(x, y) \mapsto x \cdot y$ on a poset, defined whenever $x \leq y$, such that it is monotone in each argument and $x \cdot x = x$ for every x. Then x is closed in y when $x \cdot y = x$, and dually x is dense in y when $x \cdot y = y$.

The first result in the paper is to show that bco's as defined before are a specialised instance of cco's on posets. Moreover, every cco for a class M of monomorphisms in a category, under a mild condition on M together with the requirement that M is closed under composition, gives rise to a family of bco's on posets of M-subobjects, thus allowing to deduce properties of general cco's from the corresponding bco's. This fact is very interesting and useful, since bco's are simpler that cco's, and bco's have an intrinsic duality, which cco's lack.

Deepening the connection, the bco's on a poset with finite joins such that y is closed in z whenever $x \cdot z \leq y \leq z$, are exactly those arising from unary interior operators (the dual of unary closure operators). Moreover, idempotence of a bco have several equivalent characterisations, from which one derives a strong link of bco's with factorisation systems and torsion theory.

In the overall, the paper is apparently technical in its results, however, the clean presentation and the elegant mathematical development makes it appealing not only for the specialist, but also for mathematicians interested in the deep links among the algebra of orders, topology, and categories.

Reviewer: Marco Benini (Buccinasco)

MSC:

18A32 Factorization systems, substructures, quotient structures, congruences, amalgams
06A15 Galois correspondences, closure operators (in relation to ordered sets)
18B35 Preorders, orders, domains and lattices (viewed as categories)
06C05 Modular lattices, Desarguesian lattices

Keywords:

closure operator; factorization system; modular law; torsion theory

Full Text: DOI

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