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Some approximation properties and nuclear operators in spaces of analytical functions. (English) Zbl 1404.46017

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Recall that a Banach space X has the approximation property (AP) if the identity operator in X can be approximated by finite rank operators in the topology of uniform convergence on compact sets. Since every compact set is contained in the closed absolutely convex hull of a sequence which converges to zero, the AP can be reformulated as follows: A Banach space X has the AP if, for every sequence $(x_n)_n \subset X$ with $\|x_n\| \longrightarrow 0$ as $n \to \infty$ and $\varepsilon > 0$, there exists a finite rank operator R in X such that $\sup_{n\in\mathbb{N}} \|x_n - Rx_n\| < \varepsilon$.

The authors take this last form of the AP and introduce a weak version of the AP: Denote by $\mathcal{C}$ the set of all real positive sequences tending to ∞. Then, for a sequence $c = (c_n)_n \in \mathcal{C}$, a Banach space X has the approximation property with respect to c $(AP_{[c]})$ if, for every $\varepsilon > 0$ and any sequence $(x_n)_n$ in X with $\|x_n\| \leq c_n^{-1}$, there exists a finite rank operator R in X such that $\|Rx_n - x_n\| \leq \varepsilon$ for every n. Also, if $\mathcal{C}_0$ is a subset of $\mathcal{C}$, X has the approximation property respect to $\mathcal{C}_0$ $(AP_{[\mathcal{C}_0]})$ if it has the $AP_{[c]}$ for all $c \in \mathcal{C}_0$.

These type of approximation properties generalize some AP's considered in different works. For instance, in [*J. Bourgain* and *O. Reinov*, Math. Nachr. 122, 19–27 (1985; Zbl 0584.46039)], it was shown that H^∞ (the space of bounded analytic functions in the unit disk) has the $AP_{[\log(n)]}$ and, if $\ell_p^{-1} = \{(c_n)_n \in \mathcal{C} : (c_n^{-1})_n \in \ell_p\}$, then the $AP_{\ell_p^{-1}}$ coincide with the AP_s considered, among others, in [*O. I. Reinov*, J. Math. Anal. Appl. 415, No. 2, 816–824 (2014; Zbl 1323.47020)], where $\frac{1}{s} = 2 - \frac{1}{p}$. Also, note that the $AP_{[\mathcal{C}]}$ is exactly the AP.

Several characterizations of the $AP_{[c]}$ are given and some results are applied to L_p-spaces, to H^∞ and its predual. Also, the authors give a nice approach in the study of whether the nuclear operators are a regular operator ideal or not, which is one of the main results of the article. Let me explain this a little:

Recall that an operator $S : X \to Y$ is nuclear if there exist sequences $(x'_n)_n \subset X'$ and $(y_n)_n \subset Y$ with $\sum_{n=1}^\infty \|x'_n\|\|y_n\| < \infty$ such that $S = \sum_{n=1}^\infty x'_n \otimes y_n$. Let $J_Y : Y \to Y''$ be the natural injection of Y into its second dual and take an operator $T : X \to Y$. If $J_Y \circ T : X \to Y''$ is nuclear, then must T be nuclear? When X' or Y''' has the AP, the answer is yes and the hypothesis over X or Y is sharp.

Here, the authors introduce, for a sequence $c \in \mathcal{C}$, the $[c]$-nuclear operators from X to Y as those T which admit a representation of the form $Tx = \sum_{n=1}^\infty \mu_n x'_n(x) y_n$ for $x \in X$, where $(x'_n)_n \subset X'$ and $(y_n)_n \subset Y$ with $\sum_{n=1}^\infty \|x_n\|\|y_n\| < \infty$ and $|\mu_n| \leq 1/c_n$. Then, they show that an operator $T : X \to Y$ such that $J_Y \circ T : X \to Y''$ is $[c]$-nuclear is nuclear itself if X' or Y''' has the $AP_{[c]}$.

Reviewer: Pablo Turco (Buenos Aires)

MSC:

46B28 Spaces of operators; tensor products; approximation properties
47B10 Linear operators belonging to operator ideals (nuclear, p-summing, in the Schatten-von Neumann classes, etc.)

Keywords:

nuclear operator; tensor product; approximation property; space of bounded analytical functions

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