Towers, John D.

Convergence via OSLC of the Godunov scheme for a scalar conservation law with time and space flux discontinuities. (English) Zbl 1395.65042

Numer. Math. 139, No. 4, 939-969 (2018).

Summary: This paper deals with a Godunov scheme as applied to a scalar conservation law whose flux has discontinuities in both space and time. We extend the definition of vanishing viscosity solution of *K. H. Karlsen* and *J. D. Towers* [J. Hyperbolic Differ. Equ. 14, No. 4, 671–701 (2017; Zbl 1380.65158)] (which applies to a flux with a spatial discontinuity) in order to accommodate the addition of temporal flux discontinuities, and prove that this extended definition implies uniqueness. We prove convergence of the Godunov approximations to the unique vanishing viscosity solution as the mesh size converges to zero, thus establishing well-posedness for the problem. The novel aspect of this paper is the use of a discrete one-sided Lipschitz condition (OSLC) in the discontinuous flux setting. In the classical setting where flux discontinuities are not present, the OSLC is well known to produce an immediate regularizing effect, with a local spatial variation bound resulting at any positive time. We show that the OSLC also produces a regularizing effect at any finite distance from the spatial flux discontinuity. This regularizing effect is not materially affected by temporal flux discontinuities. When combined with a Cantor diagonal argument, these regularizing effects imply convergence of the Godunov approximations. With this new method it is possible to forgo certain assumptions about the flux that seem to be required when using two commonly used convergence methods.

MSC:

65M08 Finite volume methods for initial value and initial-boundary value problems involving PDEs

65M12 Stability and convergence of numerical methods for initial value and initial-boundary value problems involving PDEs

35L65 Hyperbolic conservation laws

Full Text: DOI

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