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The nested simple conformal loop ensembles in the Riemann sphere. (English) Zbl 1352.60117

Probab. Theory Relat. Fields 165, No. 3-4, 835-866 (2016).

The paper under review studies various basic concepts in conformal field theory, from explicit probabilistic constructions and interpretations through nested CLE (Conformal Loop Ensembles), where a CLE in a simply connected planar domain D is a random countable collection of simple loops that are all contained in D. The laws of CLEs are characterized by Markovian exploration described by *S. Sheffield* and the second author [Ann. Math. (2) 176, No. 3, 1827–1917 (2012; Zbl 1271.60090)]. CLEs as the collections of outer boundaries of outermost clusters in Poissonian collections of Brownian loops in D and the loops in a CLE that are very closely related to SLE_k curves for $k \in (8/3, 4]$ are described by *O. Schramm* [Isr. J. Math. 118, 221–288 (2000; Zbl 0968.60093)]. There is one CLE_k law for each k.

The first main result (Theorem 1) states that for any $k \in (8/3, 4]$ the law of the nested CLE_k in the full complex plane is invariant under $z \mapsto 1/z$, and the other main result derives and highlights properties of this particular full-plane structure with intensity measures $\nu^i = \nu^e = \alpha \times \nu^{cle}$ for some constant $\alpha = \alpha(k)$, where $\nu^i(A)$ (ν^e or ν^{cle}) is the mean number of loops inside (outside or surrounding the origin) which are contained in a measurable set. Using inversion invariance of the $\mathrm{SLE}_{3/8}$ loop soup, the first main result is deduced from the second main result.

Section 2 reviews loops and clusters from the $\mathrm{SLE}_{8/3}$ loop soup. The Brownian loop soup $\beta = \{\beta_j : j \in J\}$ in C with intensity c is a Poisson point process in the plane with intensity $c\mu$, where J is the set of all points appearing in the point process. Sheffield and the second author [loc. cit.] showed that Brownian loop-soup clusters in D are disjoint for $c \leq 1$ ($c \in (0, 1]$ corresponds to $k(c) \in (8/3, 4]$). The second author [J. Am. Math. Soc. 21, No. 1, 137–169 (2008; Zbl 1130.60016)] showed that the family $\eta = \{\eta_j, j \in J\}$ of outer boundaries of outermost clusters is a Poisson point process of $\mathrm{SLE}_{8/3}$ loops, and this random family is invariant under any Möbius transformation of the Riemann sphere (in particular $z \mapsto 1/z$).

Subsection 2.2 discusses Markov chains of nested clusters and of nested loops. The scale invariance of the loop soup (the expected number of clusters that surround the origin and have diameter between 1 and 2 is finite) provides three infinite measures ν, ν^i, ν^e as the intensity measure of the families K_j, γ_j^i, γ_j^e, where K_j is a cluster that disconnects infinity from the origin, the boundaries of the complement K_j consist of countably many loops γ_j^i that contain the origin, and γ_j^e contains infinity in the noncompact component. Proposition 1 states that ν is invariant under $z \mapsto 1/z$ and its proof follows from the invariance of the full-plane $\mathrm{SLE}_{8/3}$ loop soup under inversion. Subsection 2.3 tries to decompose the information provided by a loop γ that surrounds the origin into its size and its shape.

Section 3 describes the construction of the full-plane CLEs. The authors first construct a coupling between nested CLEs for two simply connected domains D and D' that surround the origin (Subsection 3.1) by showing (Proposition 2) that there almost surely exists an n_0 such that the two nested CLEs coincide inside the loop γ_{n_0}. The full-plane nested CLE chain to b from ∞ is the restriction of the full-plane nested CLE to those loops that surround b, thus ν^{cle} is defined to be the infinity intensity measure of CLE ($\infty \to 0$). Proposition 3 states that $\nu^e = \nu^i$. By checking that ν^i is invariant under the kernel $Q^{\to e}$, Theorem 1 follows from Proposition 3 with the invariance in distribution under the map $z \mapsto 1/z$ of the nested family CLE($\infty \to 0$) of loops γ_j, $j \in J$. Subsection 3.4 extends the previous result to the annular region (no longer simply connected) with two controlled parameters (m_1, m_2).

Section 4 is devoted to the proof of $\nu^i = \nu^e$. Sheffield and the second author [loc. cit.] constructed a CLE by starting from a Poisson point process of SLE_k bubbles, and Subsection 4.1 reviews their ideas and tools. The proof starts to build measures as $\varepsilon \to 0$, and, via interchanging the order of integration and by inversion, provides expressions for both ν_ε^i and ν_ε^e. The expressions have Euclidean areas with key asymptotic values (Lemma 3), and the other terms can be identified.

Subsection 4.2 is devoted to the proof of Lemma 3. A similar result on the Minkowski content of chordal SLE paths (not loops) was proved by *G. F. Lawler* and *M. A. Rezaei* [Ann. Probab. 43, No. 3, 1082–1120 (2015; Zbl 1331.60165)]. Using analogous arguments, one considers now instead of a chordal SLE path an SLE bubble defined under the infinite measure μ and keeps the function J as before, then restricts to the study of loops in the plane that surround the origin and stay confined in a given annulus. One of the main ingredients in the proofs of Lawler and Rezaei [loc. cit., Theorem 3.1] is to gain a very good control on the probability that the SLE intersects the ε-neighborhood of a point in the upper half-plane. The second main ingredient in their proof is to control the second moments of $\nu_{\varepsilon,J}$ and their variation with respect to ε, where

$$\nu_{\varepsilon,J}(\beta) = \int_H d^2 z J(z) 1_{d(z,\beta)<\varepsilon}.$$

The hitting time of a disc around z by the chordal $\mathrm{SLE}_k\beta$ follows from Sections 2.3 and 4.2 of [Lawler and Rezaei, loc. cit.].

It would be nice to see the main results in the paper under review to have applications to the critical Ising model from statistical physics and the bulk stress-energy tensor from conformal field theory.

Reviewer: Weiping Li (Stillwater)

MSC:

60J67 Stochastic (Schramm-)Loewner evolution (SLE)
60D05 Geometric probability and stochastic geometry
60G55 Point processes (e.g., Poisson, Cox, Hawkes processes)

Cited in **9** Documents

Keywords:

random curves; conformal loop ensembles; Schramm-Loewner evolution; loop soups; conformal invariance; Poisson point process; nested cluster; scaling limit

Full Text: DOI arXiv

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