Roschy, Helmut; Rahn, Thorsten
Cohomology of line bundles: proof of the algorithm. (English) Zbl 1314.55013
J. Math. Phys. 51, No. 10, 103520, 11 p. (2010).

Summary: We present a proof of the algorithm for computing line bundle valued cohomology classes over toric varieties conjectured by *R. Blumenhagen* et al. [ibid 51, No. 10, 103525, 15 p. (2010; Zbl 1314.55012)] and suggest a kind of Serre duality for combinatorial Betti numbers that we observed when computing examples.

MSC:

55R20	Spectral sequences and homology of fiber spaces in algebraic topology	Cited in **1** Review
14M25	Toric varieties, Newton polyhedra, Okounkov bodies	Cited in **9** Documents
55U10	Simplicial sets and complexes in algebraic topology	
05E45	Combinatorial aspects of simplicial complexes	

Full Text: DOI arXiv

References:

[1] Blumenhagen, R., Jurke, B., Rahn, T., and Roschy, H., "Cohomology of line bundles: A computational algorithm" (2010); ; Blumenhagen, R., Jurke, B., Rahn, T., and Roschy, H., "Cohomology of line bundles: A computational algorithm" (2010); ; · Zbl 1314.55012

[2] The speed-optimized implementation in C++ can be downloaded from and is regularly updated. To get a first experience of the calculations possible, one can also have a quick start with a short Mathematica script that is also available there.

[3] Blumenhagen, R., Jurke, B., Rahn, T., and Roschy, H., "Cohomology of line bundles: Applications" (unpublished). · Zbl 1273.81180

[4] Grayson, D. and Stillman, M., "Macaulay 2, a software system for research in algebraic geometry," Available by ftp at .

[5] In order to do sheaf cohomology computations on general toric varieties, the additional package "NORMALTORICVARIETIES.M2" written by Smith, G. is needed. Since this is still work in progress, it is not yet included in the official distribution, but the package content can be copied from his homepage, , and then separately loaded into MACAULAY2.

[6] Cox, D. A.; Little, J. B.; Schenck, H., Toric Varieties

[7] Cvetic, M., Garcia-Etxebarria, I., and Halverson, J., "Global F-theory models: Instantons and gauge dynamics" (2010); ; Cvetic, M., Garcia-Etxebarria, I., and Halverson, J., "Global F-theory models: Instantons and gauge dynamics" (2010); ; · Zbl 1214.81149

[8] Note that it can be shown that Čechcohomology on an open cover of a toric variety can be shown to be isomorphic to sheaf cohomology, see Theorem 9.0.4 in Ref. 6.

[9] Jow, S. -Y., "Cohomology of toric line bundles via simplicial Alexander duality"; ; Jow, S. -Y., "Cohomology of toric line bundles via simplicial Alexander duality"; ; · Zbl 1315.55010

[10] In the sense of Chap. 3 of Ref. 6.

[11] A condensed introduction to simplicial complexes meeting our requirements is given, e.g., by the first chapter of Ref. 15.

[12] For a short review of sheaf theory and sheaf cohomology have a look at the appendix of Ref. 1.

[13] Note that we always identify Picard group and class group of X, since in the smooth case all Weil divisors are already Cartier.

[14] The shift in the rank comes from a shift between the ordinary and the local Čech complex, see also Theorem 9.5.7 in Ref. 6.

[15] Miller, E.; Strumfels, B., Combinatorial Commutative Algebra, 227, (2005), Springer: Springer, New York

[16] Here, the term "power set of an ideal" stands for taking all possible unions of the generators. In fact, the sequences for remnant cohomology in the algorithm of Ref. 1 come from the combinatorics of this power set and the connection with the full Taylor resolution of S / I will be important for the proof.

[17] For example, the Taylor resolution of the Stanley-Reisner ring of $X = d P_3$ is not minimal, since the subset consisting of $m_1 = x_1 x_2$, $m_2 = x_1 x_3$, and $m_3 = x_2 x_3$ is among the generators of its Stanley-Reisner ideal, cf. the examples in Ref. 1.

[18] See Ref. 25 for more details on these categorical issues.

[19] Eisenbud, D.; Mustaţă, M.; Stillman, M., J. Symb. Comput., 29, 583, (2000) · Zbl 1044.14028 · doi:10.1006/jsco.1999.0326

[20] We want to note that the set (α, σ) corresponds to all so-called \textit{rationoms}$\mathbf{x}^{\mathbf{u}}$ with $f(\mathbf{u}) = \alpha$ and precisely the coordinates x_i with $i ⊠ \sigma$ standing in the denominator. Intuitively,

these rationoms can be interpreted as “representatives” of Čech cohomology on intersections of open sets in the toric variety, cf. Sec. 2.2 of Ref. 1.

[21] This corresponds to the sequences for “remnant cohomology” in Ref. 1. By counting the number of times that a fixed denominator $\mathbf{x}^\sigma$ appears in rank r of the Stanley-Reisner power set, one gets the number of $(r - 1)$-faces of the complex Γ^σ. If one also takes notice of the different combinations of Stanley-Reisner generators that lead to this denominator, one can write down the maps in Eq. (40) and gets a well-defined complex.

[22] MacLagan, D.; Smith, G., J. Reine Angew. Math., 571, 179, (2004)

[23] Bayer, D.; Charalambous, H.; Popescu, S., J. Algebra, 221, 497, (1999) · Zbl 0946.13008 · doi:10.1006/jabr.1999.7970

[24] For $X = d P_3$ the divisor $D = - 3 H - X - Y - Z$ that we took as an example in Ref. 1 is such a boundary divisor with the same charge as $x_1^{- 1} x_2^{- 1} x_3^{- 1}$.

[25] Weibel, C., An Introduction to Homological Algebra, 38, (1994), Cambridge University Press: Cambridge University Press, Cambridge, England · Zbl 0797.18001