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Flowability of plane homeomorphisms. (English. French summary) Zbl 1296.37032

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It is well known that every fixed-point free homeomorphism of the real line is the time-one map of some flow, i.e., it is embeddable into a flow (or briefly, "flowable"). The paper under review studies flowability of planar Brouwer (fixed-point free orientation-preserving) homeomorphisms that preserve the leaves of a Reeb foliation. The latter foliation is defined as follows: there is an invariant straight strip whose complement is foliated by parallel lines; the strip is foliated by parabola-like curves going to infinity in just one prescribed direction.

The boundary of the strip consists of two straight-line leaves Δ and Δ'. The main result of the paper is Theorem 1.1, which yields a flowability criterion for a homeomorphism $f:\mathbb{R}^2\to\mathbb{R}^2$ in terms of the equivalence relation $\simeq_f$ on $\Delta\times\Delta'$ generated by f.

By definition, $(x,x')\simeq_f(y,y')$, if there exist sequences $x_k\to x$, $y_k\to y$ such that $f^k(x_k)\to x'$ and $f^k(y_k)\to y'$. Theorem 1.1 is a very interesting result; it says that f is flowable, if and only if the latter equivalence relation satisfies just two very simple and explicit isochronicity matching conditions: the four-point matching property and the eight-point matching property.

The main part of Theorem 1.1 says that each homeomorphism satisfying the above matching properties is flowable. To prove it, the authors first construct the flow on the Reeb strip boundary and then extend it to the whole plane. It is well known that every fixed-point free line homeomorphism $f:\mathbb{R}\to\mathbb{R}$ ($\mathbb{R}\simeq\Delta,\Delta'$) can be embedded into infinitely many different flows. The first step of the proof (Proposition 2.1) shows that each flow is uniquely defined by its induced equivalence relation on $\mathbb{R}\times\mathbb{R}$: $(x_1,y_1)\equiv(x_2,y_2)$ if (x_2,y_2) is a time-t flow image of the pair (x_1,y_1) for some t. The key part of the proposition provides a simple and very explicit criterion for an equivalence relation to be induced by a flow: a flowability criterion.

In the next step, the authors show that the above equivalence relation $\simeq_f$ induces an equivalence relation on $\Delta\times\Delta$ that satisfies the above flowability criterion. This allows to prescribe a flow generating f on $\Delta\cup\Delta'$. Then it is extended to the whole plane. The continuity of the extended flow on the closure of the strip is ensured by the four-point matching property. At the end of the paper, the authors show that both the four-and eight-point matching properties are necessary for flowability.

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MSC:

37E30 Dynamical systems involving homeomorphisms and diffeomorphisms of planes and surfaces

37E35 Flows on surfaces

Keywords:

Brouwer homeomorphism; flow; foliation; homeomorphism; plane; Reeb component

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