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The biharmonic Neumann problem in Lipschitz domains. (English) Zbl 1216.35021

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From the introduction: Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain with connected boundary $\partial\Omega$. The main purpose of this article is to solve the following Neumann problem for the biharmonic equation in Lipschitz domains:

$$\Delta^2 u = 0, \tag{1}$$

$$\nu \Delta u + (1-\nu)\frac{\partial^2 u}{\partial N^2} = f_0, \tag{2}$$

$$\frac{\partial \Delta u}{\partial N} + (1-\nu)\frac{1}{2}\frac{\partial}{\partial T_{ij}}\left(\frac{\partial^2 u}{\partial N \partial T_{ij}}\right) = \Lambda_0. \tag{3}$$

Here f_0 is prescribed in an appropriate Lebesgue space $L^p(\partial\Omega)$ with respect to the surface measure ds, Λ_0 is a linear functional prescribed in the dual space to the Sobolev space $W^{1,p'}(\partial\Omega)$ with respect to the surface measure, and ν is a constant known as the Poisson ratio. A unique solution u (modulo linear functions) is obtained in the class of solutions with nontangential maximal function of the second-order derivatives in $L^p(\partial\Omega)$.

N denotes the outer unit normal vector to the domain, and T various tangential directions to the Lipschitz boundary. The components of these vectors are not better than bounded measurable functions. If the Poisson ratio takes the value 1, the problem is not well-posed. Consequently the second- and third-order directional derivatives in (2) are always present. The second-order derivatives, formed with the Hessian matrix for u, do not include differentiations of the component functions. The third directional differentiation does, in the sense of distributions.

The quantities in the boundary operators depend only on the local Lipschitz geometry of the domain. Because this geometry is measured in a scale-invariant way, estimates for the Neumann problem (1) and (2) must also be scale invariant if they are to depend only on the geometry. Moreover, the boundary operators are independent of any particular choice of orientation for the rectangular coordinate system. The boundary operators are intrinsic to the geometry of the boundary.

In this article solutions are shown to exist with derivatives up to second order that converge pointwise nontangentially a.e. (ds) and in $L^p(\partial\Omega)$. The third-order data is shown to converge in the sense of distributions (using parallel approximating boundaries) in the space $W^{-1,p}(\partial\Omega)$ dual to $W^{1,p'}(\partial\Omega)$. The analysis here is basically $p = 2$, but a perturbation of all estimates to a small interval about $p = 2$ is shown to depend only on the Lipschitz geometry of the domain, and solvability there also follows. The optimal range for p, which from known results will also depend on dimension, must be investigated elsewhere.

MSC:

35J40 Boundary value problems for higher-order elliptic equations
35Q74 PDEs in connection with mechanics of deformable solids

Full Text: DOI

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