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On quasilinear Brezis-Nirenberg type problems with weights. (English) Zbl 1209.35055

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The authors consider radially symmetric positive solutions of the p-Laplacian problem

$$\begin{cases} -\Delta_p u = \lambda C(|x|)|u|^{p-2}u + B(|x|)|u|^{q-2}u, & \text{in } B_R(0)\setminus\{0\}, \\ u = 0 & \text{on } \partial B_R(0), \\ \lim_{|x|\to 0} |x|^{N-1}|\nabla u(x)|^{p-1} = 0, \end{cases} \tag{1}$$

where $B_R(0)$ denotes, for $R > 0$, the open ball in $\mathbb{R}^N$ of radius R, centered at 0. It is assumed that $q \ge p > 1$, $N \ge p$, and that the weights B, C are positive and such that $r^{N-1}B(r)$ and $r^{N-1}C(r)$ are in $L^1(0,R)$.

First a critical exponent p^* is defined that depends on B and p and controls the compactness of embeddings of certain weighted Sobolev spaces. The existence of a smallest eigenvalue λ_1, which is positive, and a corresponding eigenfunction φ_1 for the nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda C(|x|)|u|^{p-2}u & \text{in } B_R(0)\setminus\{0\}, \\ u = 0 & \text{on } \partial B_R(0), \\ \lim_{|x|\to 0} |x|^{N-1}|\nabla u(x)|^{p-1} = 0, \end{cases} \tag{2}$$

is proved under appropriate conditions on p and C.

With these notions, and imposing additional assumptions on the weights B and C and the exponents p and q, the authors prove existence and nonexistence results roughly of the following form: The subcritical problem (1), where $q < p^*$, has a nontrivial, radially symmetric, and nonnegative solution if and only if $\lambda < \lambda_1$. In the critical case $q = p^*$, if p and q satisfy a condition that restricts the dimension N, there exist $0 < \lambda^* < \lambda^{**} < \lambda_1$ such that (1) has a nontrivial, radially symmetric, and nonnegative solution if $\lambda \in (\lambda^{**}, \lambda_1)$, and it has no such solution if $\lambda \in (0, \lambda^*)$. Finally, (1) has no such solution if q is supercritical, i.e. $q > p^*$, and R and $\lambda > 0$ are small enough.

The dimension restriction in the critical case exhibits certain *critical dimensions*, similarly as in the celebrated result of *H. Brézis* and *L. Nirenberg* [Commun. Pure Appl. Math. 36, 437–477 (1983; Zbl 0541.35029)].

Reviewer: Nils Ackermann (Mexico City)

MSC:

35J92	Quasilinear elliptic equations with p-Laplacian	
35J20	Variational methods for second-order elliptic equations	
35J25	Boundary value problems for second-order elliptic equations	
35B09	Positive solutions to PDEs	
35B07	Axially symmetric solutions to PDEs	
35B33	Critical exponents in context of PDEs	

Keywords:

Brezis-Nirenberg type results; critical exponent; nonlinear eigenvalue; positive radial solutions; quasilinear elliptic equation; weight function