Matoušek, Jiří; Tancer, Martin; Wagner, Uli
Hardness of embedding simplicial complexes in $\mathbb{R}^d$. (English) Zbl 1208.68130
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Summary: Let $\mathrm{EMBED}_{k\to d}$ be the following algorithmic problem: Given a finite simplicial complex K of dimension at most k, does there exist a (piecewise linear) embedding of K into $\mathbb{R}^d$? Known results easily imply polynomiality of $\mathrm{EMBED}_{k\to 2}$ ($k=1,2$; the case $k=1, d=2$ is graph planarity) and of $\mathrm{EMBED}_{k\to 2k}$ for all $k\geq 3$.

We show that the celebrated result of Novikov on the algorithmic unsolvability of recognizing the 5-sphere implies that $\mathrm{EMBED}_{d\to d}$ and $\mathrm{EMBED}_{(d-1)\to d}$ are undecidable for each $d\geq 5$. Our main result is NP-hardness of $\mathrm{EMBED}_{2\to 4}$ and, more generally, of $\mathrm{EMBED}_{k\to d}$ for all k,d with $d\geq 4$ and $d\geq k\geq (2d-2)/3$. These dimensions fall outside the metastable range of a theorem of Haefliger and Weber, which characterizes embeddability using the deleted product obstruction. Our reductions are based on examples, due to Segal, Spiez, Freedman, Krushkal, Teichner, and Skopenkov, showing that outside the metastable range, the deleted product obstruction is not sufficient to characterize embeddability.

MSC:

68Q17 Computational difficulty of problems (lower bounds, completeness, difficulty of approximation, etc.)
57Q35 Embeddings and immersions in PL-topology

Keywords:

algorithmic unsolvability; finite simplicial complex; (piecewise linear) embeddability; NP-hard

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