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The Schrödinger operator with Morse potential on the right half-line. (English)

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Motivated by the search of a spectral representation of the zeros of Riemann's zeta function via hypothetical "Hilbert-Polya operators", the author considers, for real k and the Morse potential

$$V_k(u) := \frac{1}{4}\, e^{2u} + ke^u,$$

the Schrödinger operator $H_k := -(d/du)^2 + V_k$ on the half-line $[u_0, \infty)$, using various boundary conditions at u_0. Denote by $W_{\kappa,\mu}$ the solution of Whittaker's equation

$$\left(\frac{d^2}{dx^2} + \left(-\frac{1}{4} + \frac{\kappa}{x} + \frac{1/4 - \mu^2}{x^2}\right)\right) f(x) = 0$$

that decays as $x \to \infty$ and set

$$Z_1(z) := W_{-k,z-1/2}(e^{u_0}).$$

Then Z_1 is an entire function of $z \in \mathbb{C}$ and z is a zero of Z_1 if and only if $-(z-1/2)^2$ is an eigenvalue of H_k with Dirichlet boundary conditions. Since in this case H_k is self adjoint and positive in $L^2([u_0, \infty))$, it follows that all zeros of Z_1 lie on the line $1/2 + it$, $t \in \mathbb{R}$. The function Z_1 may therefore be thought of as a toy model of Riemann's zeta function, and its zero distribution can be studied via the correspondence with the eigenvalues of H_k. Moreover, there is also a deeper correspondence at the level of de Branges spaces and canonical systems.

As an example for the types of results obtained in this way, using the asymptotic density of eigenvalues of H_k, it is shown that the total number $N(T)$ of zeros z with $|\mathrm{Im}(z)| \leq T$ satisfies the asymptotic formula

$$N(T) = \frac{2}{\pi}\, T \log T + \frac{2}{\pi}\, (2\log 2 - 1 - u_0)T + O(1)$$

as $T \to \infty$, where the $O(1)$-term depends on k.

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MSC:

34L40 Particular ordinary differential operators (Dirac, one-dimensional Schrödinger, etc.)
34L15 Eigenvalues, estimation of eigenvalues, upper and lower bounds of ordinary differential operators
11M26 Nonreal zeros of $\zeta(s)$ and $L(s, \chi)$; Riemann and other hypotheses

Keywords:

Riemann's Zeta function; asymtotic density of eigenvalues; asymtotic density of zeros; de Branges spaces; canonical systems

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