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A family of numerical schemes for kinematic flows with discontinuous flux. (English)

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Summary: Multiphase flows of suspensions and emulsions are frequently approximated by spatially one-dimensional kinematic models, in which the velocity of each species of the disperse phase is an explicitly given function of the vector of concentrations of all species. The continuity equations for all species then form a system of conservation laws which describes spatial segregation and the creation of areas of different composition. This class of models also includes multi-class traffic flow, where vehicles belong to different classes according to their preferential velocities. Recently, these models were extended to fluxes that depend discontinuously on the spatial coordinate, which appear in clarifier-thickener models, in duct flows with abruptly varying cross-sectional area, and in traffic flow with variable road surface conditions. This paper presents a new family of numerical schemes for such kinematic flows with a discontinuous flux. It is shown how a very simple scheme for the scalar case, which is adapted to the "concentration times velocity" structure of the flux, can be extended to kinematic models with phase velocities that change sign, flows with two or more species (the system case), and discontinuous fluxes. In addition, a MUSCL-type upgrade in combination with a Runge-Kutta-type time discretization can be devised to attain second-order accuracy. It is proved that two particular schemes within the family, which apply to systems of conservation laws, preserve an invariant region of admissible concentration vectors, provided that all velocities have the same sign. Moreover, for the relevant case of a multiplicative flux discontinuity and a constant maximum density, it is proved that one scalar version converges to a BV_t entropy solution of the model. In the latter case, the compactness proof involves a novel uniform but local estimate of the spatial total variation of the approximate solutions. Numerical examples illustrate the performance of all variants within the new family of schemes, including applications to problems of sedimentation, traffic flow, and the settling of oil-in-water emulsions.

MSC:

76M12	Finite volume methods applied to problems in fluid mechanics
76T20	Suspensions

Keywords:

conservation law; discontinuous flux; kinematic flow; numerical scheme; traffic model

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