Coates, Tom; Givental, Alexander
Quantum Riemann-Roch, Lefschetz and Serre. (English) Zbl 1189.14063
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Using the formalism of quantized quadratic Hamiltonians [*A. B. Givental*, Mosc. Math. J. 1, No. 4, 551–568 (2001; Zbl 1008.53072)], the authors are able to prove quantum versions of three classical theorems in algebraic geometry; namely, the Riemann-Roch theorem, Serre duality, and the Lefschetz hyperplane section theorem. The key ingredient consists in introducing a notion of twisted Gromov-Witten invariants of a compact projective complex manifold X; the quantum version of the aforementioned theorems can then be seen as relations between the twisted and the nontwisted Gromov-Witten theory of X.

More precisely, let $X_{g,n,d}$ be the moduli space of genus g, n-pointed stable maps to X of degree d, where d is an element in $H_2(X;\mathbb{Z})$, and let E be a holomorphic vector bundle on X. Since a point in $X_{g,n,d}$ is represented by a pair (Σ, f), where Σ is a complex curve and $f:\Sigma\to X$ a holomorphic map, one can use f to pull back E on Σ and then consider the K-theory Euler character of f^*E, i.e., the virtual vector space $H^0(\Sigma, f^*E)\ominus H^1(\Sigma, f^*E)$, as the fiber over $[(\Sigma, f)]$ of a virtual vector bundle $E_{g,n,d}$ over $X_{g,n,d}$. This intuitive construction is made completely rigorous by considering K-theory push-pull $K^0(X)\to K^0(X_{g,n,d})$ along the diagram

$$\begin{array}{ccc} X_{g,n+1,d} & \xrightarrow{\mathrm{ev}_{n+1}} & X \\ {\scriptstyle\pi}\downarrow & & \\ X_{g,n,d} & & \end{array}$$

A rational invertible multiplicative characteristic class of a complex vector bundle is an expression of the form

$$\mathbf{c}(\cdot) = \exp\left(\sum_{k=0}^{\infty} s_k \mathrm{ch}_k(\cdot)\right),$$

where ch_k are the components of the Chern character, and the s_k are arbitrary parameters. These data determine a cohomology class $\mathbf{c}(E_{g,n,d})$ (actually, a formal family of cohomology classes parametrized by the s_k) in $H^*(X_{g,n,d};\mathbb{Q})$, and one can define the total $(\mathbf{c},E)$-twisted descendant potential $\mathcal{D}^g_{\mathbf{c},E}$ as

$$\mathcal{D}_{\mathbf{c},E}(t_0,t_1,\dots) = \exp\left(\sum_{g\geq 0} \hbar^{g-1}\mathcal{F}^g_{\mathbf{c},E}(t_0,t_1,\dots)\right),$$

where

$$\mathcal{F}^g_{\mathbf{c},E}(t_0,t_1,\dots) = \sum_{n,d}\frac{Q^d}{n!}\int_{[X_{g,n,d}]}\mathbf{c}(E_{g,n,d})(\sum_{k_1=0}^{\infty}(\mathrm{ev}_1^* t_k)\psi_1^{k_1})\cdots(\sum_{k_1=0}^{\infty}(\mathrm{ev}_n^* t_k)\psi_n^{k_n}).$$

Here Q^d is the representative of d in the semigroup ring of degrees of holomorphic curves in X, $t_0, t_1, \dots$ are rational cohomology classes on X, and ψ_i is the first Chern class of the universal cotangent bundle over $X_{g,n,d}$ corresponding to the i-th marked point of X. For E the zero element in $K^0(X)$, the twisted potential $\mathcal{D}^g_{\mathbf{c},E}$ reduces to $\mathcal{D}_X$, the total descendant potential of X.

At this point the formalism of quantized quadratic hamiltonians enters the picture. One considers the symplectic space $\mathcal{H} = H^*(X;\mathbb{Q})((z^{-1}))$ of Laurent polynomials in z^{-1} with coefficients in the cohomology of X, endowed with the symplectic form

$$\Omega(\mathbf{f},\mathbf{g}) = \frac{1}{2\pi i}\oint\left(\int_X \mathbf{f}(-z)\mathbf{g}(z)\right)dz.$$

The subspace $\mathcal{H}_+ = H^*(X;\mathbb{Q})[z]$ is a Lagrangian subspace, and $(\mathcal{H},\Omega)$ is identified with the canonical symplectic structure on $T^*\mathcal{H}_+$. Finally, given an infinitesimal symplectic transformation T of $\mathcal{H}$, one can consider the differential operator $\hat{T}$ of order ≤ 2 on functions on $\mathcal{H}_+$, which is associated by quantization to the quadratic Hamiltonian $\Omega(T\mathbf{f},\mathbf{f})/2$ on $\mathcal{H}$. By the inclusion $\mathcal{H}_+\hookrightarrow H^*(X;\mathbb{Q})[[z]]$, the operator $\hat{T}$ acts on asymptotic elements of the Fock space, i.e., on functions of the formal variable $\mathbf{q}(z) = q_0 + q_1 z + q_2 z^2 + \cdots$ in $H^*(X;\mathbb{Q})[[z]]$. By the dilaton shift, i.e., setting $\mathbf{q}(z) = \mathbf{t}(z) - z$, with $\mathbf{t}(z) = t_0 + t_1 z + t_2 z^2 + \cdots$, the operator $\hat{T}$ acts on any function of $t_0, t_1, \ldots$, notably on the descendant potentials.

Having introduced this formalism, the authors are able to express the relation between twisted and untwisted Gromov-Witten invariants in an extremely elegant way: up to a scalar factor,

$$\mathcal{D}_{\mathbf{c},E} = \hat{\Delta}\mathcal{D}_X,$$

where $\Delta : \mathcal{H} \to \mathcal{H}$ is the linear symplectic transformation defined by the asymptotic expansion of

$$\sqrt{\mathbf{c}(E)} \prod_{m=1}^{\infty} \mathbf{c}(E \otimes L^{-m})$$

under the identification of the variable z with the first Chern class of the universal line bundle L. This is the quantum Riemann-Roch theorem; it explicitly determines all twisted Gromov-Witten invariants, of all genera, in terms of untwisted invariants. The result is a consequence of Mumford's Grothendieck-Riemann-Roch theorem applied to the universal family $\pi : X_{g,n+1,d} \to X_{g,n,d}$. If $E = \mathbb{C}$ is the trivial line bundle, then $E_{g,n,d} = \mathbb{C} \ominus \mathbf{E}_g^*$, where $\mathbf{E}_g$ is the Hodge bundle, and one recovers from quantum Riemann-Roch results of *D. Mumford* [Arithmetic and geometry, Pap. dedic. I. R. Shafarevich, Vol. II: Geometry, Prog. Math. 36, 271–328 (1983; Zbl 0554.14008)] and *C. Faber, R. Pandharipande* [Invent. Math. 139, No.1, 173–199 (2000; Zbl 0960.14031)] on Hodge integrals.

If $\mathbf{c}^*$ is the multiplicative characteristic class

$$\mathbf{c}^*(\cdot) = \exp\left(\sum_{k=0}^{\infty}(-1)^{k+1} s_k \mathrm{ch}_k(\cdot)\right),$$

then $\mathbf{c}^*(E^*) = 1/\mathbf{c}(E)$, and one the following quantum version of Serre duality:

$$\mathcal{D}_{\mathbf{c}^*,E^*}(\mathbf{t}^*) = (\mathrm{sdet}\,\mathbf{c}(E))^{-\frac{1}{24}}\mathcal{D}_{\mathbf{c},E}(\mathbf{t}),$$

where $\mathbf{t}^*(z) = \mathbf{c}(E)\mathbf{t}(z) + (1 - \mathbf{c}(E))z$.

Finally, if E is a convex vector bundle and a submanifold $Y \subset X$ is defined by a global section of E, then the genus zero Gromov-Witten invariants of Y can be expresssed in terms of the invariants of X twisted by the Euler class of E. These are in turn related to the untwisted Gromov-Witten invariants of X by the quantum Riemann-Roch theorem, so the authors end up with a quantum Lefschetz hyperplane section principle, expressing genus zero Gromov-Witten invariants of a complete intersection Y in terms of those of X. This extends earlier results [*V. V. Batyrev, I. Ciocan-Fontanine, B. Kim* and *D. van Straten*, Acta Math. 184, No. 1, 1–39 (2000; Zbl 1022.14014); *A. Bertram*, Invent. Math. 142, No. 3, 487–512 (2000; Zbl 1031.14027); *A. Gathmann*, Math. Ann. 325, No. 2, 393–412 (2003; Zbl 1043.14016); *B. Kim*, Acta Math. 183, No. 1, 71–99 (1999; Zbl 1023.14028); *Y.-P. Lee*, Invent. Math. 145, No. 1, 121–149 (2001; Zbl 1082.14056)], and yields most of the known mirror formulas for toric complete intersections. The idea of deriving mirror formulas by applying the Grothendieck-Riemann-Roch theorem to universal stable maps is not new: according to the authors it can be traced back at least to Kontsevich's investigations in the early 1990s, and to Faber's and Pandharipande's work on Hodge integrals.

Reviewer: Domenico Fiorenza (Roma)

MSC:

14N35	Gromov-Witten invariants, quantum cohomology, Gopakumar-Vafa invariants, Donaldson-Thomas invariants (algebro-geometric aspects)
14C40	Riemann-Roch theorems
14J33	Mirror symmetry (algebro-geometric aspects)

Keywords:

Gromov-Witten invariants; mirror symmetry; Grothendieck-Riemann-Roch theorem; Lefschetz hyperplane section principle; Serre duality

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