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Existence and nonexistence of solutions for singular quadratic quasilinear equations. (English) Zbl 1173.35051

J. Differ. Equations 246, No. 10, 4006-4042 (2009).

The paper is concerned with quasilinear elliptic problems of the form

$$\begin{cases} -\mathrm{div}(M(x,u)\nabla u) + g(x,u)|\nabla u|^2 = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here $\Omega \subseteq \mathbb{R}^N$ is open and bounded, and $N \geq 3$. The coefficients are Caratheodory functions, and such that the principal part is uniformly elliptic with bounded coefficients. For the nonhomogeneous part f it is assumed that it lies in a suitable Lebesgue space, and that it is uniformly bounded from below by positive constants on compact subsets of Ω.

The main interest lies in nonnegative functions g with a singularity in $u = 0$ that is uniform in x. Suppose that $h : (0, \infty) \to [0, \infty)$ is continuous, nonincreasing in a neighborhood of zero, and satisfies

$$\lim_{s\to 0+} \int_s^1 \sqrt{h(t)}\, dt < \infty.$$

If

$$g(x,s) \leq h(s) \qquad \text{for a.e. } x \in \Omega,\ \forall s > 0,$$

then it is proved that (1) has a weak positive solution in $H_0^1(\Omega)$.

Conversely, a nonexistence result is given in the case that g grows faster in s than a function h whose square root is not integrable near 0. For the model problem

$$\begin{cases} -\Delta u + \frac{|\nabla u|^2}{u^\gamma} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

this amounts to the following assertion: Suppose that $\gamma > 0$. Then Eq. (2) has a positive solution if and only if $\gamma < 2$.

Existence of a solution to (1) is proved by applying classical results for quasilinear equations to a family of problems with truncated coefficients and then passing to the limit.

The regularity of solutions to (1) is also considered. Moreover, the authors treat a general semilinear variant of (1).

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MSC:

35J60 Nonlinear elliptic equations
35D05 Existence of generalized solutions of PDE (MSC2000)
35D10 Regularity of generalized solutions of PDE (MSC2000)
35B45 A priori estimates in context of PDEs

Keywords:

quasilinear elliptic equation; singular coefficient; positive solution; nonexistence

Full Text: DOI Link

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