Bouchez, Vincent; Willem, Michel
Extremal functions for the Caffarelli-Kohn-Nirenberg inequalities: A simple proof of the symmetry. (English) Zbl 1171.35041
J. Math. Anal. Appl. 352, No. 1, 293-300 (2009).

Let Ω denote an open subset of the cylinder $\mathcal{C} := S^{N-1} \times \mathbb{R}$. The authors consider symmetry of solutions to the equation

$$-\Delta_\sigma v + \lambda v = f(v), \qquad \sigma \in \Omega, \tag{1}$$

posed in $H_0^1(\Omega)$, with $\lambda \geq 0$, f continuous, and $f(0) = 0$.

Fixing $P \in S^{N-1}$, Ω is *P-symmetric* if $\Omega \cap (\mathbb{R}^N \times \{t\})$ is a geodesic ball in $S^{N-1} \times \{t\}$ with center (P,t), for every $t \in \mathbb{R}$. Suppose from now on that Ω is P-symmetric. Informally, a measurable function $v : \Omega \to \mathbb{R}$ is *foliated Schwarz symmetric with respect to P* if there is $g : [0,\pi] \times \mathbb{R} \to \mathbb{R}$ which is nonincreasing in its first argument, and such that

$$v(\theta, t) = g(\operatorname{dist}(\theta, P), t)$$

for every $(\theta, t) \in \Omega$. Here $\operatorname{dist}(\theta, P)$ denotes the geodesic distance of θ and P on S^{N-1}.

Denote $F(u) := \int_0^u f(s)\,ds$ and

$$\Phi(v) := \frac{1}{2} \int_\Omega (|\nabla v|^2 + v^2)\,d\sigma - \int_\Omega F(v)\,d\sigma,$$

for $v \in H_0^1(\Omega)$. It is proved that every solution v of (1) that minimizes Φ among all nontrivial solutions is foliated Schwarz symmetric.

The proof is very short and relies on a technique based on polarizations, as developed by one of the authors. It is then shown how this result implies foliated Schwarz symmetry for extremal functions in the Caffarelli-Kohn-Nirenberg inequalities, as previously shown in [*C.-S. Lin, Z.-Q. Wang*, Proc. Am. Math. Soc. 132, No. 6, 1685–1691 (2004); erratum ibid. 132, No. 2183 (2004; Zbl 1036.35028)].

Reviewer: Nils Ackermann (México)

MSC:

35J60 Nonlinear elliptic equations
35J20 Variational methods for second-order elliptic equations

Keywords:

Foliated Schwarz Symmetric Solution; Polarizations

Full Text: DOI

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