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Nonresonance theory for semilinear operator equations under regularity conditions. (English) Zbl 1169.47053

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Given a Hilbert space H with inner product $\langle \cdot, \cdot \rangle_H$ and a selfadjoint positive operator $A : D(A) \to H$ in H with compact resolvent, the energy space H_A of A is defined as $H_A := A^{-1/2}(H)$, with inner product $\langle u, v \rangle_{H_A} := \langle A^{1/2}u, A^{1/2}v \rangle_H$. Suppose that Y is another Hilbert space, that $S \in \mathcal{L}(H_A, Y)$ satisfies $\|S\|_{\mathcal{L}(H_A,Y)} \leq 1$, and that $F : H \times Y \to H$ is a continuous map. Then the nonlinear operator equation

$$Au = cu + F(u, Su), \qquad u \in H_A, \tag{1}$$

is considered in the nonresonant case $c \notin \sigma(A)$. Here, one looks for weak solutions of (1), i.e., elements $u \in H_A$ such that $\langle u, v \rangle_{H_A} = \langle cu + F(u, Su), v \rangle_H$ for all $v \in H_A$.

Using Banach's fixed point theorem and Leray-Schauder theory, various existence theorems are given for (1). The conditions on F amount to at most linear growth in u and v, and in some instances include (partial) global Lipschitz conditions on F. The growth and Lipschitz constants appearing here are of the order $1/\|(A - c)^{-1}\|$. In some cases, the existence of a closed subspace Z of Y is assumed that embeds compactly and satisfies $D_A \subseteq S^{-1}(Z)$. The authors call this assumption a regularity condition.

One application is the following. Let $\Omega \subseteq \mathbb{R}^N$ denote a smoothly bounded domain. Let A denote $-\Delta$ on Ω with respect to Dirichlet boundary conditions. Hence $E := L^2(\Omega)$ and $H_A := H_0^1(\Omega)$. Suppose that c is not an eigenvalue of A. Let $Y := L^2(\Omega, \mathbb{R}^N)$, $S := \nabla$, and $Z := H^1(\Omega, \mathbb{R}^N)$. Then $D(A) = H_0^1(\Omega) \cap H^2(\Omega) \subseteq S^{-1}(Z) = H^2(\Omega)$.

Suppose that $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ induces a continuous superposition operator $F : H \times Y \to H$ that satisfies

$$\|F(u, v)\|_H \leq a\|u\|_H + b\|v\|_Y + h$$

for some small enough $a, b > 0$ and some $h > 0$. Then there is at least one solution $u \in H_0^1(\Omega) \cap H^2(\Omega)$ of

$$-\Delta u = cu + f(u, \nabla u).$$

The main point seems to be that F is only bounded linearly in u and v, not necessarily sublinearly.

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MSC:

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nonlinear operator equation; non resonance condition; fixed point; Leray-Schauder theory; semilinear elliptic equation; convection term