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Compactness results and applications to some "zero mass" elliptic problems. (English)

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Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods 69, No. 10, 3559-3576 (2008).

In this paper the existence of solutions for the semilinear elliptic equation $-\Delta u = f'(u)$ in an unbounded subdomain of $\mathbb{R}^3$ is considered. Specifically, the "zero mass" case is treated, that is, $f'(0)$ and $f''(0)$ vanish.

To use the variational structure of this class of problems, the natural space to work in is $\mathcal{D}^{1,2}(\Omega)$, where $\Omega \subseteq \mathbb{R}^3$ is unbounded. One also considers the space sum $L^p + L^q(\Omega)$ with the norm

$$\|v\|_{L^p+L^q(\Omega)} := \inf\{\|v_1\|_{L^p(\Omega)} + \|v_2\|_{L^q(\Omega)} \mid (v_1, v_2) \in L^p(\Omega) \times L^q(\Omega),\ v = v_1 + v_2\}.$$

If $1 < p < 6 < q$ then it is known that there is a continuous embedding $\mathcal{D}^{1,2}(\Omega) \hookrightarrow L^p + L^q(\Omega)$.

Let Ω have suitable symmetries and denote by $\mathcal{D}^{1,2}_{\mathrm{s}}(\Omega)$ a subspace of suitably symmetric functions of $\mathcal{D}^{1,2}(\Omega)$. One of the aims of the authors is to prove the compactness of the embedding $\mathcal{D}^{1,2}_{\mathrm{s}}(\Omega) \hookrightarrow L^p + L^q(\Omega)$ in various situations. This information can be used to prove the relative compactness of Palais-Smale sequences for the application of variational methods.

In the first application, consider $f \in C^1(\mathbb{C}, \mathbb{R})$, satisfying $f(0) = 0$, $f(M) > 0$ for some $M > 0$, $|f'(\xi)| \le C \min\{|\xi|^{p-1}, |\xi|^{q-1}\}$ for some constant $C > 0$ and all $\xi \in \mathbb{C}$, and $f(\xi) = f(|\xi|)$ for all $\xi \in \mathbb{C}$. In particular, f has supercritical growth at 0 and subcritical growth at ∞. It is proved that then for every $n \in \mathbb{Z}$ the equation $-\Delta v = f'(v)$ has a complex valued solution $v^{(n)} \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ of the form $v^{(n)}(x, y, z) = u^{(n)}(r, z)\mathrm{e}^{\mathrm{i}n\theta}$, where (r, θ, z) represent cylindrical coordinates and $u^{(n)}(r, z) \in \mathbb{R}$.

The second application is the equation $-\Delta v = f'(v)$ posed on $\mathbb{R}^2 \times I$, where I is an open bounded interval in $\mathbb{R}$. Here $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies $f(0) = 0$, $f(\xi) \ge C_1 \min\{|\xi|^p, |\xi|^q\}$ and $f'(\xi) \le C_2 \min\{|\xi|^{p-1}, |\xi|^{q-1}\}$ for some positive constants C_1, C_2 and all $\xi \in \mathbb{R}$, and the weak Ambrosetti-Rabinowitz condition $\alpha f(\xi) \le f'(\xi)\xi$ with some constant $\alpha \ge 2$ and for all $\xi \in \mathbb{R}$. Then existence of infinitely many cylindrically symmetric solutions is proved.

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MSC:

35J60	Nonlinear elliptic equations
35J20	Variational methods for second-order elliptic equations
46E30	Spaces of measurable functions (L^p-spaces, Orlicz spaces, Köthe function spaces, Lorentz spaces, rearrangement invariant spaces, ideal spaces, etc.)
46E35	Sobolev spaces and other spaces of "smooth" functions, embedding theorems, trace theorems

Keywords:

semilinear elliptic problem; zero mass; compactness of embeddings

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References:

[1] Almgren, F.J.; Lieb, E.H., Symmetric decreasing rearrangement is sometimes continuous, J. amer. math. soc., 2, 683-773, (1989) · Zbl 0688.46014

[2] Ambrosetti, A.; Felli, V.; Malchiodi, A., Ground states of nonlinear Schrödinger equations with potentials vanishing at infinity, J. eur. math. soc. (JEMS), 7, 117-144, (2005) · Zbl 1064.35175

[3] Ambrosetti, A.; Rabinowitz, P.H., Dual variational methods in critical point theory and applications, J. funct. anal., 14, 349-381, (1973) · Zbl 0273.49063

[4] Aubin, T., Problémes isopérimétriques et espaces de Sobolev, J. differential geom., 11, 573-598, (1976) · Zbl 0371.46011

[5] A. Azzollini, A multiplicity result for a semilinear Maxwell type equation, preprint

[6] Azzollini, A.; Benci, V.; D'Aprile, T.; Fortunato, D., Existence of static solutions of the semilinear Maxwell equations, Ricerche mat., 55, 123-137, (2006)

[7] Badiale, M.; Rolando, S., Elliptic problems with singular potential and double-power nonlinearity, Mediterr. J. math., 4, 417-436, (2005) · Zbl 1121.35050

[8] Bartolo, P.; Benci, V.; Fortunato, D., Abstract critical point theorems and applications to some nonlinear problems with "strong" resonance at infinity, J. nonlinear anal. TMA, 7, 981-1012, (1983) · Zbl 0522.58012

[9] Benci, V.; Fortunato, D., Towards a unified theory for classical electrodynamics, Arch. ration. mech. anal., 173, 379-414, (2004) · Zbl 1065.78004

[10] Benci, V.; Micheletti, A.M., Solutions in exterior domains of null mass nonlinear field equations, Adv. nonlinear stud., 6, 171-198, (2006) · Zbl 1114.35143

[11] Benci, V.; Grisanti, C.R.; Micheletti, A.M., Existence and non existence of the ground state solution for the nonlinear Schrödinger equations with $V(\infty) = 0$, Topol. methods nonlinear anal., 26, 203-219, (2005) · Zbl 1105.35112

[12] Benci, V.; Grisanti, C.R.; Micheletti, A.M., Existence of solutions for the nonlinear Schrödinger equations with $V(\infty) = 0$, (), 53-65 · Zbl 1231.35225

[13] Benci, V.; Visciglia, N., Solitary wave with non-vanishing angular momentum, Adv. nonlinear stud., 3, 151-161, (2003) · Zbl 1030.35051

[14] Berestycki, H.; Lions, P.L., Nonlinear scalar field equations. I. existence of a ground state, Arch. ration. mech. anal., 82, 313-345, (1983) · Zbl 0533.35029

[15] Berestycki, H.; Lions, P.L., Nonlinear scalar field equations. II. existence of infinitely many solutions, Arch. ration. mech. anal., 82, 347-375, (1983) · Zbl 0556.35046

[16] Berestycki, H.; Lions, P.L., Existence d'états multiples dans des équations de champs scalaires non linéaires dans le cas de masse nulle, C. R. acad. sci. Paris Sér. I math., 297, 267-270, (1983) · Zbl 0542.35072

[17] Caffarelli, L.A.; Gidas, B.; Spruck, J., Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth, Comm. pure appl. math., 42, 3, 271-297, (1989) · Zbl 0702.35085

[18] Esteban, M.J.; Lions, P.L., A compactness lemma, Nonlinear anal. TMA, 7, 381-385, (1983) · Zbl 0512.46035

[19] M. Ghimenti, A.M. Micheletti, Existence of minimal nodal solutions for the nonlinear Schroedinger equations with $V(\infty) = 0$, preprint · Zbl 1146.35412

[20] Krasnosel'skii, M.A.; Rutickii, Y.B., Convex functions and Orlicz spaces, (1961), P. Noordhoff Ltd Groningen

[21] Lions, P.L., Minimization problems in $L^1(\mathbb{R}^3)$, J. funct. anal., 41, 236-275, (1981) · Zbl 0464.49019

[22] Lions, P.L., Symétrie et compacité dans LES espaces de Sobolev, J. funct. anal., 49, 315-334, (1982) · Zbl 0501.46032

[23] Lions, P.L., Solutions complexes d'équations elliptiques semilinéaires dans $\mathbb{R}^N$, C. R. acad. sci. Paris S. I math., 302, 673-676, (1986) · Zbl 0606.35027

[24] L. Pisani, Remarks on the sum of Lebesgue spaces, preprint

[25] Talenti, G., Best constant in Sobolev inequality, Ann. mat. pura appl., 110, 353-372, (1976) · Zbl 0353.46018