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The Lie theory of connected pro-Lie groups. A structure theory for pro-Lie algebras, pro-Lie groups, and connected locally compact groups. (English) [Zbl 1153.22006]

This monograph provides a systematic study of pro-Lie groups, a class of topological groups which contains all connected locally compact groups. The authors are well-known experts who have been working in Lie theory and the theories of compact and locally compact groups for decades, and who already published a related monograph on compact groups [The structure of compact groups, de Gruyter, Berlin (1998; Zbl 0919.22001; second edition 2006; Zbl 1139.22001)].

The objects of study are pro-Lie groups, i.e., complete topological groups $G$ which can be approximated by Lie groups in the sense that each identity neighbourhood $U \subseteq G$ contains a closed normal subgroup $N \subseteq G$ such that $G/N$ is a (finite-dimensional, real) Lie group. By a classical theorem of H. Yamabe [Ann. Math. (2) 58, 351–365 (1953; Zbl 0053.01602)], every connected locally compact group is a pro-Lie group, and more generally every locally compact group $G$ which is almost connected in the sense that the factor group $G/G_0$ of $G$ modulo the connected component $G_0$ of the identity is compact. Moreover, every compact group is a pro-Lie group and also every locally compact abelian group. Hence all of the latter are projective limits of Lie groups, and this makes techniques and results from Lie theory amenable to the study of locally compact groups.

It is clear from the preceding examples that pro-Lie groups form a rich and relevant class of topological groups (henceforth denoted PL), and it turns out that this is also a pleasant and gratifying class from a category-theoretical point of view: PL is closed under the formation of closed subgroups and direct products, and hence closed under the formation of arbitrary limits (in the category $\text{TG}$ of topological groups). By contrast, the category $\text{LCG}$ of locally compact groups does not go along well with projective limits: Neither is it closed in $\text{TG}$ under the formation of such, nor do projective limits need to exist within $\text{LCG}$.

Because the very gadget needed to link locally compact groups with Lie groups is not well-adapted to this class of groups, it is a natural idea to try to develop, instead, a systematic theory of pro-Lie groups – within which results concerning locally compact groups are then obtained as special cases. The book under review makes a very convincing case for this strategy. It becomes clear that the investigation of pro-Lie groups is not only interesting because of the added generality, but unavoidable for intrinsic reasons, and relevant also for those who are primarily interested in locally compact groups.

To substantiate this claim, let us recall that it is a highly successful strategy in finite-dimensional Lie theory to prepare results on the level of Lie algebras, transfer them to the associated simply connected Lie groups and finally to pass to the case of general connected Lie groups. This strategy stops to work in the category of locally compact groups because the natural analogues of simply connected Lie groups (namely projective limits of simply connected Lie groups) frequently fail to be locally compact. However, it carries over to the realm of pro-Lie groups (albeit not without difficulty), and it is in keeping to this strategy that the authors are able to obtain their deep structural results.

The subject matter brings about that methods from category theory need to be used to some extent. However, a working knowledge of basic concepts like epics, direct products, limits and pairs of adjoint functors suffices, and many of the essential ideas are recalled to assist less experienced readers. Also the prerequisites concerning topological groups and Lie theory are moderate.

It should be mentioned that a proof of Yamabe’s Theorem (as well as the solution to Hilbert’s Fifth Problem) are not part of the book under review – for these results, which are easy to take on faith, the reader is referred to classical sources like the monograph [D. Montgomery and L. Zippin, Topological transformation groups. Interscience Publishers, New York (1955; Zbl 0068.01904)]. Rather, the book focusses on new, complementary material, and Yamabe’s Theorem is only needed to ensure that almost connected locally compact groups are among the objects under consideration.

The book begins with an extended (62 page) “panoramic overview,” which provides an introduction to the topic and surveys the main results obtained in the book. Besides the bibliography and appendices...
(which subsume an introduction to Lie groups), the book is structured into 14 chapters:

1. Limits of topological groups  
2. Lie groups and the Lie theory of topological groups  
3. Pro-Lie groups  
4. Quotients of pro-Lie groups  
5. Abelian pro-Lie groups  
6. Lie’s third fundamental theorem  
7. Profinite-dimensional modules and Lie algebras  
8. The structure of simply connected pro-Lie groups  
9. Analytic subgroups and the Lie theory of pro-Lie groups  
10. The global structure of connected pro-Lie groups  
11. Splitting theorems for pro-Lie groups  
12. Compact subgroups of pro-Lie groups  
13. Iwasawa’s local splitting theorem  
14. Catalog of examples.

Before I return to a discussion of the book in general terms, let me give a summary of its contents:

Chapters 1 and 2 are of a preparatory nature. Chapter 1 compiles some background from category theory (notably concerning limits and adjoint functors) and basic results concerning projective limits of topological groups. Given a topological group $G$, let $L(G)$ be the set of all continuous group homomorphisms (morphisms) from $(\mathbb{R}, +)$ to $G$, equipped with the compact-open topology. Chapter 2 begins with a definition of (Banach-) Lie groups (which is explained further in an appendix) and then extends various Lie-theoretic concepts to the class of topological groups $G$ for which $L(G)$ is a topological Lie algebra in a canonical fashion (with addition and Lie bracket determined by the Trotter product formula and commutator formula, respectively). For example, the evaluation map $\exp_G: L(G) \to G, X \mapsto X(1)$ is an appropriate analogue of the exponential function of a Lie group. Every projective limit of Lie groups is a topological group with Lie algebra. Because the functor $L$ is compatible with limits, it takes a projective limit $G$ of Lie groups $G_i$ (which we always assume finite-dimensional) to the corresponding projective limit of the finite-dimensional Lie algebras $L(G_i)$. Thus $L(G)$ is a projective limit of finite-dimensional Lie algebras, a so-called pro-Lie algebra. Conversely, there is a certain analogue of Lie’s Third Theorem: If a pro-Lie algebra $g$ is a projective limit of finite-dimensional Lie algebras $g_i$, one can associate to it the projective limit $\Gamma(g)$ of the simply connected Lie groups $\Gamma(g_i)$ with Lie algebra $g_i$. Then $L(\Gamma(g)) \cong g$, and it is not hard to see that $\Gamma(g)$ is a pro-Lie group. If $G$ is a connected pro-Lie group, then $\tilde{G} := \Gamma(L(G))$ is a certain substitute for the universal covering group of a Lie group, together with a natural morphism $\pi_G: \tilde{G} \to G$ (which, however, need not even be surjective). Once the stage is set, the presentation of beautiful results can begin. The main result of Chapter 3 is the Pro-Lie Group Theorem (Theorem 3.34), which shows that pro-Lie groups and projective limits of Lie groups coincide. As a consequence, closed subgroups of pro-Lie groups are pro-Lie, and moreover the class of pro-Lie groups is closed in the category of topological groups under arbitrary limits. It also becomes clear at this point that the exponential image generates a dense subgroup of the identity component of a pro-Lie group (proof of Lemma 3.22). The proofs essentially use that $L(G)$ is a so-called weakly complete topological vector space, i.e., isomorphic to $\mathbb{R}^J$ for some set $J$.

Chapter 4 is devoted to quotients of pro-Lie groups. If $G$ is a pro-Lie group and $N$ a closed normal subgroup of $G$, then the topological quotient group $G/N$ is a pro-Lie group if and only if it is complete (which is not always the case, as the authors recall from a joint paper with D. Poguntke [Forum Math. 16, No. 1, 1–16 (2004; Zbl 1041.22005)]. Various criteria for completeness of $G/N$ are given. Irrespective of completeness, the following “One Parameter Subgroup Lifting Lemma” is obtained: Every morphism $X: \mathbb{R} \to G/N$ arises as $q \circ Y$ for some $Y \in L(G)$, where $q: G \to G/N$ is the canonical quotient morphism.

Chapter 5 is devoted to abelian pro-Lie groups, and contains results concerning both the structure of such groups, and Pontryagin duality theory for them. Part of these results generalize familiar facts from the theory of locally compact groups: For example, there is a version of Weil’s lemma ensuring that a morphism from $\mathbb{R}$ to a pro-Lie group either is a topological embedding or has relatively compact image.
Also, there is a Vector Group Splitting Theorem. It deals with so-called vector group complements $V$ in
an abelian pro-Lie group $G$, i.e., subgroups $V$ of $G$ such that $V$ is isomorphic to the additive group of
a weakly complete topological vector space and such that $G_0$ is the internal direct product of its subgroup
of compact elements and $V$. The theorem ensures that $G = V \times H$ internally for a suitable subgroup $H$.
The authors then show that every identity neighbourhood in an abelian pro-Lie group $G$ contains a cotorus
subgroup $D$ of $G$ (i.e., $G/D$ is a torus and the quotient morphism $G \to G/D$ induces an isomorphism
$L(G) \to L(G/D)$). For $V$ and $H$ as before, one can now choose a cotorus subgroup $D$ of $H$. Then $D$ is
pro-discrete (i.e., a projective limit of discrete groups) and $\delta: L(G) \times D \to G$, $(X,y) \mapsto \exp_G(x)y$ is a
quotient morphism with pro-discrete kernel (Resolution Theorem). Moreover, setting $\Delta := D \cap H_0$, there
are quotient morphisms

$$L(G) \times D \to G \to (G_0/\Delta) \times (D/\Delta)$$

with pro-discrete kernels (Sandwich Theorem). For some purposes, $\delta$ is a better substitute for a (in
general non-existing) universal covering morphism than $\pi_G$. Many of the preceding results are adapted
to non-abelian groups later.

Chapter 6 discusses some technical aspects concerning pro-simply connected pro-Lie groups (i.e., projective
limits of simply connected Lie groups) and the functor $g \mapsto \Gamma(g)$.

To reach a more detailed understanding of the structure of pro-Lie groups, the authors follow the general
strategy of Lie theory, and begin (in Chapter 7) with a refined study of the structure of pro-Lie algebras. If
$g$ is a pro-Lie algebra, then it is a projective limit of finite-dimensional $g$-modules under the adjoint action,
and the topological dual space $g'$ is a union of finite-dimensional $g$-modules (under the coadjoint action).
Via duality, decompositions of $g'$ as a direct sum of submodules correspond to direct product decompo-
sitions of $g$. Following this basic idea, the authors succeed in reducing structural questions concerning $g$
to purely algebraic questions concerning the module $g'$. For example, they characterize suitably-defined
reductive pro-Lie algebras as direct products of finite-dimensional simple or one-dimensional Lie algebras.
Likewise, semisimple pro-Lie algebras are characterized as direct products of finite-dimensional simple
Lie algebras. The authors also give various characterizations of pro-Lie algebras which are pro-solvable
(i.e., a projective limit of finite-dimensional solvable Lie algebras) or pro-nilpotent. Every pro-Lie algebra
$g$ has a radical (a largest pro-solvable ideal). The high point of Chapter 7 is the Levi-Mal’cev Theorem
for Pro-Lie Algebras. It ensures that any such is the internal semi-direct product of its radical and a
semisimple closed subalgebra, which is unique up to a special automorphism.

The preceding results concerning pro-Lie algebras have consequences for the associated pro-simply con-
nected pro-Lie groups (drawn in Chapter 8). In particular, any pro-simply connected pro-Lie group $G$ is
a semidirect product of a closed normal subgroup whose Lie algebra is the radical of $L(G)$, and a closed
subgroup whose Lie algebra is a Levi complement.

Chapter 9 compiles various technical results concerning analytic subgroups of a pro-Lie group $G$, i.e.,
images of morphisms from connected pro-Lie groups to $G$. It turns out that every connected pro-Lie
group has a largest compact central subgroup (Theorem 9.50). Moreover, there is a version of the Open
Mapping Theorem for pro-Lie groups: If $q: G \to H$ is a surjective morphism between pro-Lie groups and
$G$ is almost connected, then $q$ is open.

Chapters 10–13 are devoted to the structure of general connected (or almost connected) pro-Lie groups.
They contain some of the deepest results, which depend on the foundations patiently laid in the preceding
chapters.

Chapter 10 begins with characterizations of connected pro-Lie groups which are pro-solvable (i.e.,
projective limits of solvable finite-dimensional Lie groups). Every connected pro-Lie group $G$ contains a
largest connected pro-solvable normal subgroup $R(G)$, its so-called radical; the latter is closed and its Lie
algebra is the radical of $L(G)$ (cf. Theorem 10.25). Moreover, $G/R(G)$ is a pro-Lie group and semisimple
(i.e., it has trivial radical). Various characterizations of connected pro-Lie groups $G$ are given which are
semisimple or reductive (i.e., $R(G)$ is contained in the centre of $G$).

Chapter 11 is devoted to various splitting results. The authors first discuss situations when a connected
pro-Lie group splits as a semidirect product of its commutator subgroup and an abelian subgroup (The-
orem 11.8). They then show that normal, weakly complete vector subgroups of a pro-Lie group with
compact quotient always have a semidirect complement (Theorem 11.15), and prove the uniqueness of
complements up to conjugacy (Theorem 11.31). They also draw consequences concerning the structure
of almost connected pro-solvable pro-Lie groups (Theorem 11.28).

Chapter 12 is devoted to compact subgroups of pro-Lie groups. It is shown that every connected pro-Lie

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group contains a largest compact normal subgroup (Theorem 12.59). Also, every compact subgroup of a connected pro-Lie group $G$ is contained in a maximal compact subgroup $C$, any two maximal compact subgroups are conjugate, and each of them is connected (Theorem 12.77). Moreover, $G$ is homeomorphic to a direct product of $C$ and some weakly complete topological vector space, by a suitable topological splitting result (Theorem 12.81). Among other things, also a version of the Resolution Theorem for connected pro-Lie groups is obtained.

The book culminates in Chapter 13, which provides an analogue of Iwasawa’s local splitting theorem for pro-Lie groups: Let $G$ be a pro-Lie group of finite nildimension (this condition means that $L(G)$ is a direct product of a finite-dimensional Lie algebra and a reductive pro-Lie algebra). Then every identity neighbourhood of $G$ contains a closed, normal, almost connected subgroup $N$ such that $G/N$ is a Lie group and the groups $G$ and $N \times G/N$ are locally isomorphic. Examples show that the conclusion can go wrong if $G$ does not have finite nildimension. If $G$ is locally compact, then $G$ automatically has finite nildimension and one recovers (if $G$ is connected) K. Iwasawa’s classical theorem [see Ann. Math. (2) 50, 507–558 (1949; Zbl 0034.01803)].

Summing up, the book contains a thorough treatment of the structure of connected pro-Lie groups, and achieves a multitude of deep and impressive results.

It should be mentioned that, although the book is a research monograph, it has many qualities of a textbook, and is well-suited for self-study. The book is clearly written, and one always feels the authors’ passion to assist (and educate) the reader. Thus, new concepts are usually illustrated by examples, and various exercises are posed to the reader (some of which are supplied with hints). Moreover, each chapter ends with a discussion section called “postscript” which summarizes the main ideas and provides a larger perspective. Because the cross-referencing is quite complete and the reader is sometimes even reminded of earlier definitions, the reviewer hardly ever felt the need to consult the index.

Also the bibliography is reasonably complete (although one might add the quite unknown note [A. Borel, C. R. Acad. Sci., Paris 230, 1127–1128 (1950; Zbl 0036.15603)], and attention is drawn to high points in the theory of locally compact groups, like the solution to Hilbert’s Fifth Problem and Iwasawa’s Theorem. However, many other references are given only in quite general terms (notably to research in the second half of the 20th century). This sometimes makes it difficult to distinguish which results were previously known; which are new for general pro-Lie groups, but were known in the locally compact case; and which are entirely new. For example, the existence of maximal compact connected subgroups in locally compact groups was first proved by K. Iwasawa (in the article already cited) – which does not become clear.


Of course, misprints cannot entirely be avoided in such a long text (the reviewer found one in two pages or so). But most of them are easily corrected and cannot possibly create confusion (the more so because the presentation is so detailed and clear). If clarification is required, the reader may consult the list of errata provided on the web page


Unfortunately, one real error sneaked in: Part (iv) (and (iii)) of Theorem 1.34 (the Closed Subgroup Theorem for Projective Limits) does not hold as stated, which affects the proof of the Pro-Lie Group Theorem (Theorem 3.34) and a small number of later proofs. In the meantime, this error and all of its consequences could be repaired by the authors [see J. Lie Theory 18, No. 2, 383–390 (2008; Zbl 1148.22002)]. The reader is advised to read this article in parallel with Chapter 3. (An alternative route to the Pro-Lie Group Theorem adapted to the book has been described by the reviewer [J. Lie Theory 17, No. 4, 899–902 (2007; Zbl 1154.22004]). Compare also [A. A. George Michael, J. Lie Theory 16, No. 2, 221–224 (2006; Zbl 1099.22001]) for a short proof of the theorem, which however relies on a sophisticated technical result from the solution to Hilbert’s Fifth Problem.)

Despite this grain of salt, the book under review contains excellent mathematics and is a well-written and unique source of information on the structure of pro-Lie groups and connected locally compact groups. I highly recommend it to all researchers and graduate students who would like to understand the structure of connected locally compact groups and more general topological groups which can be accessed using results and ideas from Lie theory. It certainly deserves a place on the shelf (or desk) of every researcher in the area of topological groups, and can also be recommended to all with an interest in the wider horizon.
of Lie theory.

Reviewer: Helge Glöckner (Paderborn)

MSC:

22D05 General properties and structure of locally compact groups
17B65 Infinite-dimensional Lie (super)algebras
22A05 Structure of general topological groups
22E65 Infinite-dimensional Lie groups and their Lie algebras: general properties
58B25 Group structures and generalizations on infinite-dimensional manifolds

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