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On the nonhomogeneous Neumann problem with weight and with critical nonlinearity in the boundary. (English) Zbl 1133.35050

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For a given $N \geq 3$, a bounded domain $\Omega \subseteq \mathbb{R}^N$ with smooth boundary, $p \in H^1(\Omega) \cap C(\overline{\Omega})$, $Q \in C(\partial\Omega)$, p, Q being positive functions, $f \in (H^{-1}(\Omega) \cap C(\overline{\Omega}))\backslash\{0\}$, and a real parameter λ, the author considers weak solutions to the problem

$$\begin{cases} -\operatorname{div}(p(x)\nabla u) = \lambda u + f(x), & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = Q(x)|u|^{q-2}u, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where ν is the outer unit normal to $\partial\Omega$. The exponent $q = \frac{2(N-1)}{N-2}$ is critical for the trace embedding of $H^1(\Omega)$ into $L^q(\partial\Omega)$. The general assumptions for the existence results are the following: to exclude solutions that vanish identically on $\partial\Omega$ it is always assumed that the problem

$$\begin{cases} -\operatorname{div}(p(x)\nabla u) = \lambda u + f(x), & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

has no solution. Moreover, it is assumed that there is a point $x_0 \in \partial\Omega$ such that Ω locally lies on one side of the tangent hyperspace of $\partial\Omega$ at x_0, that the mean curvature with respect to ν is positive in x_0; the function p/Q^{N-2} achieves its minimum over $\partial\Omega$ at x_0, that p and Q are differentiable at x_0 with derivative equals to 0.

Theorem 1: Suppose that $\lambda < 0$ and that $\|f\|_{H^{-1}(\Omega)}$ is sufficiently small (but not 0). Under additional regularity assumptions on p, Q, and f Eq. (1) has at least two weak solutions. Denote by $0 = \lambda_1 < \lambda_2 < \lambda_3 < \dots$ the distinct eigenvalues of $L := \operatorname{div}(p\nabla\,\cdot)$ with respect to Neumann boundary condition. Fix $k \geq 2$ and denote by E_k^- the generalized eigenspace of L corresponding to $\{\lambda_1, \lambda_2, \dots, \lambda_{k-1}\}$. Let E_k^+ be its orthogonal complement in $H^1(\Omega)$ (the author does not specify the used scalar product; most likely, it is the H^1-product with p-weight in the gradient term).

Theorem 2: Suppose that $\lambda \in (\lambda_{k-1}, \lambda_k)$ and $f \in E_k^+$. If $N = 3, 4$, in addition suppose that $f(x_0) \neq 0$. Then (1) has a weak solution. The author mentions that (2) has no solutions for a function f with constant sign on Ω, but his argument is unclear. Moreover, the latter property never holds under the conditions of Theorem 2. Hence, no information is given about how large the set of data is that satisfies the conditions of Theorem 2.

Reviewer: Nils Ackermann (México, D.F.)

MSC:

35J65 Nonlinear boundary value problems for linear elliptic equations
35J20 Variational methods for second-order elliptic equations

Keywords:

weighted Laplacian; critical nonlinear Neumann boundary condition; Ekeland's variational principle; mountain pass theorem; linking theorem; min-max principle; concentration-compactness principle

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References:

[1] Aubin, J.P.; Ekeland, I., Applied nonlinear analysis, pure and applied mathematics, (1984), Wiley Interscience Publications

[2] Adimurthi, G. Mancini, in: Ambrosetti, Marino (Eds.), The Neumann problem for elliptic equation with critical non linearity, 65th Birthday of Prof. Prodi, Scoula Normale Superiore, Pisa, 1991 · Zbl 0836.35048

[3] Ambrosetti, A.; Rabinowitz, P., Dual variational methods in critical point theory and applications, J. funct. anal., 14, 349-381, (1973) · Zbl 0273.49063

[4] Adimurthi; Yadava, S.L., Positive solution for Neumann problem with critical nonlinearity on boundary, Comm. partial differential equations, 16, 1733-1760, (1991) · Zbl 0780.35036

[5] Brezis, H.; Lieb, L., A relation between pointwise convergence of functions and convergence of functionals, Proc. amer. math. soc., 88, 486-490, (1983) · Zbl 0526.46037

[6] H. Brezis, L. Nirenberg, A minimization problem with critical exponent and non-zero data, in: Symmetry in Nature, Scuola Superiore Pisa, vol. 1, 1989, pp. 129-140 · Zbl 0763.46023

[7] Cherrier, P., Problèmes de Neumann non linéaires sur LES variètès riemanniennes, J. funct. anal., 57, 154-206, (1984) · Zbl 0552.58032

[8] Comte, M.; Knaap, M.C., Solutions of elliptic equations involving critical Sobolev exponents with Neumann boundary conditions, Manuscripta math., 69, 43-70, (1990) · Zbl 0717.35029

[9] Chabrowski, J.; Yang, J., Sharp Sobolev inequality involving a critical nonlinearity on a boundary, Topol. method. nonlinear anal., 25, 1, 135-155, (2005) · Zbl 1094.35026

[10] Escobar, J.F., Sharp constant in a Sobolev trace inequality, Indiana univ. math. J., 37, 687-698, (1988) · Zbl 0666.35014

[11] Lions, P.L., The concentration – compactness principle in the calculus of variations, the limit case, part 1 and part 2, Rev. mat. iberoamericana, 1, 1, 145-201, (1985), (2) (1985) 45-121 · Zbl 0704.49005

[12] Tarantello, G., On nonhomogeneous elliptic equations involving critical Sobolev exponent, Ann. inst. H. Poincarè, 3, 281-304, (1992) · Zbl 0785.35046

[13] Zhu, M., Sharp Sobolev inequality with interior norm, Calc. var. partial differential equations, 8, 27-43, (1999) · Zbl 0918.35029

[14] Willem, M., Minimax theorem, Progr. nonlinear differential equations appl., (1996)