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Relaxation of solitons in nonlinear Schrödinger equations with potential. (English)

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Summary: We study dynamics of solitons in the generalized nonlinear Schrödinger equation with an external potential in all dimensions except for 2. For a certain class of nonlinearities such an equation has solutions which are periodic in time and exponentially decaying in space, centered near different critical points of the potential. We call those solutions which are centered near the minima of the potential and which minimize energy restricted to $\mathcal{L}^2$-unit sphere, trapped solitons or just solitons.

We prove, under certain conditions on the potentials and initial conditions, that trapped solitons are asymptotically stable. Moreover, if an initial condition is close to a trapped soliton then the solution looks like a moving soliton relaxing to its equilibrium position. The dynamical law of motion of the soliton (i.e. effective equations of motion for the soliton's center and momentum) is close to Newton's equation but with a dissipative term due to radiation of the energy to infinity.

MSC:

35Q55 NLS equations (nonlinear Schrödinger equations)
37K45 Stability problems for infinite-dimensional Hamiltonian and Lagrangian systems
81R12 Groups and algebras in quantum theory and relations with integrable systems

Keywords:

Gross-Pitaevskii equations; solitons; stability; Fermi golden rule; trapped solitons

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