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Riesz transform and perturbation. (English) Zbl 1122.58014

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Suppose that $A\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is measurable and satisfies the ellipticity and boundedness estimate

$$C^{-1}|v|^2 \le v^T A(x) v \le C|v|^2$$

with some $C > 0$. Let H denote the form closure of the divergence form operator $-\mathrm{div}A\nabla$ on $L^p(\mathbb{R}^d)$. The Riesz transform of H is the operator $\nabla H^{-1/2}$ on L^p.

It is known that the Riesz transform of H is bounded for $p \in (1, 2+\varepsilon)$, where $\varepsilon > 0$ depends on d and C. The present article gives a perturbation type criterion to extend this result to higher values of p and to closed noncompact Riemannian manifolds. The first result pertains to the Euclidean case. Suppose one is given two divergence form operators H_0 and H, induced by the corresponding matrix functions A_0 and A as above. Consider H as a perturbation of H_0, assuming that $A_0 - A \in L^q$ for some $q \in [1,\infty)$. If the Riesz transform of H_0 is bounded in L^{p_0} for some $p_0 > 2$, and if $\nabla(I+H)^{-1/2}$ (the local Riesz transform of H) is bounded in L^p for all $p \in (2, p_0)$, then the Riesz transform of H is bounded in L^p for all $p \in (2, p_0)$. A generalization of this result to weighted L^p spaces, where the weights are positive, bounded, and bounded away from 0, leads to a theorem for a noncompact, connected smooth manifold M of dimension d: Suppose that G_0 and G are two Riemannian metrics on M with uniformly equivalent associated norms on the tangent spaces. Assume that G is a L^q-perturbation of G_0 in a specified sense, for some $q \in [1,\infty)$. Denote by H_0 and H the positive Laplace operators associated with G_0 and G. Moreover, assume the norm estimate

$$\|e^{-tH_0}\|_{\mathcal{L}(L^1,L^\infty)} \le C_1 t^{-C_2}, \qquad t \ge 1,$$

with positive constants C_1, C_2. Then there holds a theorem similar to the result in $\mathbb{R}^d$, only differing in the additional assumption that the Riesz transform of H is bounded in $L^{p'}$ for $p' \in (p_0', 2)$, where p_0' is the conjugate exponent of p_0.

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MSC:

58J37 Perturbations of PDEs on manifolds; asymptotics
42B20 Singular and oscillatory integrals (Calderón-Zygmund, etc.)
47F05 General theory of partial differential operators
47B44 Linear accretive operators, dissipative operators, etc.

Keywords:

Riesz Transform; Noncompact Closed Riemannian Manifold; Perturbation

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