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Deformation theorems on non-metrizable vector spaces and applications to critical point theory. (English) Zbl 1117.58007

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Consider a Hilbert space E that is the direct sum of two infinite dimensional Hilbert spaces E^- and E^+. Denote $u = u^- + u^+$ with $u^\pm \in E^\pm$ if $u \in E$. Suppose that $\Psi \in C^2$ satisfies $\Psi(0) = 0$, $\Psi'(0) = 0$, $\Psi''(0) = 0$ and $\lim_{\|u\|_E \to \infty} \Psi(u)/\|u\|_E^2 = \infty$. Suppose moreover that Ψ is bounded below and weakly sequentially lower semicontinuous, that Ψ' is weakly sequentially continuous, and that Ψ'' is compact. Consider the variational functional $\Phi \in C^2(E)$ given by

$$\Phi(u) := \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) - \Psi(u).$$

The variational problem of finding critical points of Φ is strongly indefinite in the sense that all its critical points have infinite Morse index. Moreover, the application of degree theory to obtain Palais-Smale or Cerami sequences is delicate because the standard linking is infinite dimensional. Nevertheless, it was proved by *W. Kryszewski* and *A. Szulkin* [Adv. Differ. Equ. 3, No. 3, 441–472 (1998; Zbl 0947.35061)] and by *C. Troestler* and *M. Willem* [Commun. Partial Differ. Equations 21, No. 9–10, 1431–1449 (1996; Zbl 0864.35036)] that in fact there exists a Palais-Smale sequence at a positive level of Φ, giving rise to a nontrivial critical point of Φ under additional assumptions. The standard linking arguments are combined with the use of a different topology on E that is reminiscent of the product of the weak topology on E^- and the norm topology on E^+.

The present paper is a continuation of previous studies of the authors that aim to formalize and generalize this type of results. Only assuming $E^\pm$ to be Banach spaces, they consider above-mentioned product $\mathcal{T}$ of weak and norm topology. This new topology $\mathcal{T}$ is not metrizable, but it is induced by a family of semi-metrics, turning $(E, \mathcal{T})$ into a gage space. The authors introduce the concept of Lipschitz normality in gage spaces and give examples where this property can be verified. It is then used to construct Lipschitz partitions of unity which, in turn, are used to prove the deformation theorems necessary for a critical point theory.

In the last part some abstract critical point theorems are proved. One yields an improvement of the result of Kryszewski and Szulkin. The other two involve symmetric functionals and yield multiplicity results. Finally these theorems are applied to a spatially periodic nonlinear Dirac equation.

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MSC:

58E05	Abstract critical point theory (Morse theory, Lyusternik-Shnirel'man theory, etc.) in infinite-dimensional spaces
54E25	Semimetric spaces
35Q40	PDEs in connection with quantum mechanics

Keywords:

gage space; Lipschitz normality; deformation theorem; strongly indefinite functional; nonlinear Dirac equation

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