H. Kneser’s proof of the 4-vertex theorem and a proof by S. Mukhopadhyaya of the theorem on sextactic points of an oval lead the authors to the concept of intrinsic systems. An intrinsic system \( f \) is defined in the following way: If \( p_1, \ldots, p_n \in S^1 \) then \( f(p_1, \ldots, p_n) \) is a map from \( S^1 \) into the set of even numbers which obeys several axioms. In the case of the 4-vertex theorem for a closed curve \( C \) in the plane, \( f(p) \) is defined as follows: Consider the greatest circle enclosed by \( C \) and touching \( C \) at \( p \). Then \( f(p)q \) is the multiplicity with which the circle meets \( C \) at \( q \), and is zero if it does not meet \( C \). For an example with \( n = 2 \) let \( C \) be a strictly convex curve. Consider the greatest ellipse enclosed by \( C \) and touching \( C \) at \( p_1 \) and \( p_2 \). Then define \( f(p_1, p_2) \) similarly.

The authors work with intrinsic systems axiomatically. From their results they gain a strengthened form of the 4-vertex theorem: There are at least 4 “clean” vertices, i.e. vertices with osculating circles which do not meet \( C \) again. The theorem on sextactic points is strengthened similarly. In fact the authors also treat the case that \( C \) or \( S^1 \) is replaced by a subarc. This requires considerable effort. As already mentioned the authors use ideas of H. Kneser and S. Mukhopadhyaya, but also from the geometry of orders (0. Haupt’s contraction lemma), from S. B. Jackson and from G. Nöbeling.

The generalization referred to in the title is described in the authors’ abstract: “For a real valued periodic smooth function \( u \) on \( \mathbb{R} \), \( n \geq 1 \), one defines the osculating polynomial \( \varphi_s \) (of order \( 2n+1 \)) at a point \( s \in \mathbb{R} \) to be the unique trigonometric polynomial of degree \( n \), whose value and first \( 2n \) derivatives at \( s \) coincide with those of \( u \) at \( s \). We will say that a point \( s \) is a clean maximal flex (resp. clean minimal flex) of the function \( u \) on \( S^1 \) if and only if \( \varphi_s \geq u \) (resp. \( \varphi_s \leq u \)) and the preimage \( (\varphi - u)^{-1}(0) \) is connected.

We prove that any smooth periodic function \( u \) has at least \( n+1 \) clean maximal flexes of order \( 2n+1 \) and at least \( n+1 \) clean minimal flexes of order \( 2n+1 \).”

Furthermore, the spaces of trigonometric polynomials are replaced by more general Chebyshev spaces. A Chebyshev space of order \( 2n+1 \) has dimension at least \( 2n+1 \) and consists of \( 2\pi \)-periodic \( C^{2n} \)-functions with at most \( 2n \) zeros. The basic properties of Chebyshev spaces are described in an appendix.

Reviewer: Erhard Heil (Darmstadt)
This reference list is based on information provided by the publisher or from digital mathematics libraries. Its items are heuristically matched to zbMATH identifiers and may contain data conversion errors. It attempts to reflect the references listed in the original paper as accurately as possible without claiming the completeness or perfect precision of the matching.