Nazarov, F.; **Sodin, M.**; **Volberg, A.**
Lower bounds for quasianalytic functions. I: How to control smooth functions. (English)
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This paper is a variation on Bang's doctoral thesis concerning Denjoy-Carleman classes of C^∞-functions, published 50 years ago. The authors claim that Bang's thesis left unsufficient trace in the literature devoted to quasianalytic functions. Therefore they took liberty to reproduce some Bang's results with their proofs. Given a non-decreasing function $A:\ [1,\infty)\to(0,\infty)$, set $M_0=1$ and $M_j=M_{j-1}A(j)$, $j\geq 1$ and define the (normalized) Denjoy-Carleman class $\mathcal{C}_A([0,1])$ to be the set of $C^\infty([0,1])$-functions such that $\|f^{(j)}\|_{[0,1]}\leq M_j$, $j\in\mathbb{Z}_+$. By the classical Denjoy-Carleman theorem, the class $\mathcal{C}_A([0,1])$ is quasianalytic (i.e. it contains no non-trivial function that vanishes at a point with all derivatives) if and only if $\sum_{j=1}^\infty \frac{M_{j-1}}{MM_j}=\infty$. The Bang degree $\mathfrak{n}_f$ of $f\in\mathcal{C}_A([0,1])$ is the largest integer N such that $\sum_{\log\|f\|_{[0,1]}^{-1}<j\leq N}\frac{M_{j-1}}{M_j}<e$. The following theorems show that Bang's degree is an important characteristics of smooth functions.

Theorem A (Bang 1953). The total number of zeroes (counting with multiplicities) of $f\in\mathcal{C}_A([0,1])$ does not exceed its Bang degree.

Assume moreover that the function A is a C^1-function. Set

$$\gamma(n):=\sup_{1\leq s\leq n}\frac{sA'(s)}{A(s)}\quad\text{and}\quad\Gamma(n)=4e^{4+\gamma(n)}.$$

Theorem B. Suppose $f\in\mathcal{C}_A([0,1])$. Then for any interval $I\subset[0,1]$ and any measurable subset $E\subset I$

$$\|f\|_I\leq\left(\frac{\Gamma(2\mathfrak{n}_f)|I|}{|E|}\right)^{2\mathfrak{n}_f}\|f\|E.$$

The above inequality is in the spirit of the classical Remez and Bernstein inequalities for polynomials. Therefore it seems to be natural to ask how to band from above the Bang degree of a polynomial by its usual degree. The same question can be asked about the upper bound of the Bang degree of a real analytic function f by its Bernstein degree $\mathfrak{B}_f(K,G):=\log(\|f\|_G/\|f\|_K)$. The authors leave these questions open.

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