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Crepant resolutions of $\mathbb{C}^n/A_1(n)$ and flops of n-folds for $n = 4, 5$. (English) Zbl 1061.14052

Yui, Noriko (ed.) et al., Calabi-Yau varieties and mirror symmetry. Providence, RI: American Mathematical Society (AMS) (ISBN 0-8218-3355-3/hbk). Fields Inst. Commun. 38, 27-41 (2003).

The paper under review is intended to describe the explicit constructions of crepant resolutions of higher-dimensional orbifolds with Gorenstein quotient singularities, that is, the algebraic (or analytic) varieties such that the analytic type of each singular point is described as $\mathbb{C}^n/G$, where G is a (non-trivial) finite subgroup of the special linear group $\mathrm{SL}_n(\mathbb{C})$. For the 2-dimensional case, finite subgroups G of $\mathrm{SL}_2(\mathbb{C})$ were classically classified into ADE series. It is well known that $\mathbb{C}^2/G$ are always hypersurface singularities, and the minimal resolutions of them give the desired crepant resolutions. For the 3-dimensional case, the required crepant resolutions were found for all finite subgroups $G \subset \mathrm{SL}_3(\mathbb{C})$ by virtue of *Y. Ito* [Proc. Japan Acad., Ser. A 70, 131–136 (1994; Zbl 0831.14006)], *D. Markushevich* [Math. Ann. 308, 279–289 (1997; Zbl 0899.14016)], *S. S. Roan* [Int. J. Math. 5, 523–536 (1994; Zbl 0856.14005)] and *S. S. Roan* [Topology 35, 489–508 (1996; Zbl 0872.14034)], which are depending on the classical result on the classification of finite subgroups of $\mathrm{SL}_3(\mathbb{C})$ due to Miller-Blichfeldt-Dickson [*Y. A. Miller, H. F. Blichfeldt* and *L. E. Dickson*, "Theory and application of finite groups". New York, Wiley (1915; JFM 45.0255.12)]. But, for such a higher-dimensional case, the non-uniqueness of crepant resolutions happens due to the existence of certain kinds of codimension 2 birational operations known as flops. In order to understand higher-dimensional crepant resolutions qualitatively, the development has resulted in the theory of G-Hilbert schemes $\mathrm{Hilb}^G(\mathbb{C}^n)$ associated to the quotient singularities $\mathbb{C}^n/G$ as in *Y. Ito* and *I. Nakamura* [Proc. Japan Acad.,Ser. A 72, 135–138 (1996; Zbl 0881.14002)]: the crepant resolutions of $\mathbb{C}^n/G$ would be related to G-Hilbert schemes $\mathrm{Hilb}^G(\mathbb{C}^n)$ of G-stable 0-dimensional subschemes of $\mathbb{C}^n$ of length equal to the order $|G|$ of G. As a result, the structure of $\mathrm{Hilb}^G(\mathbb{C}^n)$ now yields that $\mathrm{Hilb}^G(\mathbb{C}^3)$ is a toric crepant resolution of $\mathbb{C}^3/G$ for every finite abelian subgroup $G \subset \mathrm{SL}_3(\mathbb{C})$ by virtue of *T. Bridgeland, A. King* and *M. Reid* [J. Am. Math. Soc. 14, 535–554 (2001; Zbl 0966.14028)], *Y. Ito* and *H. Nakajima* [Topology 39, 1155–1191 (2000; Zbl 0995.14001)] and *I. Nakamura* [J. Alg. Geom. 10, 757–779 (2001; Zbl 1104.14003)]. Whereas, for the cases $n \geq 4$, there are very few results concerning the crepant resolutions of $\mathbb{C}^n/G$ and the structure of $\mathrm{Hilb}^G(\mathbb{C})$ for finite subgroups $G \subset \mathrm{SL}_n(\mathbb{C})$. The authors restricted themselves to the case where G is the subgroup $A_r(n)$ of $\mathrm{SL}_n(\mathbb{C})$ consisting of all diagonal matrices of order $r+1$. In an earlier paper [Int.. J. Math. Math. Sci. 26, 649–669 (2001; Zbl 1065.14018)], the authors studied the case of $n = 4$ and $G = A_r(4)$, and obtained crepant resolutions of $\mathbb{C}^4/A_r(4)$ through the detailed investigations of the structure of $\mathrm{Hilb}^{A_r(4)}(\mathbb{C}^4)$. In the present paper, the authors study the case of $n = 4, 5$ and $r = 1$. More precisely, for $n = 4$ and $r = 1$, they describe the toric variety structure of $\mathrm{Hilb}^{A_1(4)}(\mathbb{C}^4)$ which is NOT crepant. Then, blowing-down the divisor $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ on $\mathrm{Hilb}^{A_1(4)}(\mathbb{C}^4)$ onto $\mathbb{P}^1 \times \mathbb{P}^1$ in different ways, they obtain three different toric crepant resolutions of $\mathbb{C}^4/A_1(4)$. These three crepant resolutions are related to each other by 4-fold flops. For the case $n = 5$ and $r = 1$ also, as in the 4-dimensional case just above, they describe the toric variety structure of $\mathrm{Hilb}^{A_1(5)}(\mathbb{C}^5)$ which is NOT crepant, and obtain twelve mutually different crepant resolutions of $\mathbb{C}^5/A_1(5)$, all of which are dominated by $\mathrm{Hilb}^{A_1(5)}(\mathbb{C}^5)$. These twelve crepant resolutions are related to each other by 5-fold flops.

For the entire collection see [Zbl 1022.00014].

Reviewer: Takashi Kishimoto (Saitama)

MSC:

14M17 Homogeneous spaces and generalizations
14E15 Global theory and resolution of singularities (algebro-geometric aspects)
14M25 Toric varieties, Newton polyhedra, Okounkov bodies
14C05 Parametrization (Chow and Hilbert schemes)

Keywords:

G-Hilbert schemes; G-clusters; toric crepant resolution