
This is a very interesting paper which gives new insight in Hilbert’s theorem. The authors prove:

Theorem 1.1: Suppose \( f(x, y, z) \) is a nonnegative real quartic form which defines a smooth plane curve \( Q = \{(x : y : z) \in \mathbb{P}^2(\mathbb{C}) : f(x, y, z) = 0\} \). Then the inequivalent representations of \( f \) as a sum of three squares (of real quadratic forms) – modulo the real orthogonal group \( O(\mathbb{R}^3) \) – are in one-to-one correspondence with the eight 2-torsion points in the non-identity component of \( J(\mathbb{R}) \), where \( J \) is the Jacobian of \( Q \).

The assumptions imply in particular: \( f \) is irreducible, \( Q(\mathbb{R}) = \emptyset \), \( Q \) has genus 3, \( J \) has 63 non-zero complex 2-torsion points. Hilbert’s original theorem (i.e. any nonnegative quartic form \( f \) is a sum of three squares in at least one way) follows from the above theorem by continuity arguments.

The proof of Theorem 1.1 proceeds in two steps:

(1) The non-trivial 2-torsion points of \( J(\mathbb{C}) \) are in one-to-one correspondence with the equivalence classes – modulo \( O(\mathbb{C}^3) \) – of representations of \( f \) as a sum of three squares of complex quadratic forms. A condensed proof using Weil divisors, the Picard group \( \text{Pic}(Q) \) and the Riemann-Roch Theorem is given in the paper. The result itself goes back to A. B. Coble (1929; JFM 55.0808.02), it was rediscovered by C. T. C. Wall (1991; Zbl 0741.14014).

(2) Show that under (1) the non-trivial 2-torsion points of \( J(\mathbb{R}) \) correspond to “signed quadratic representations” \( f = \pm q_1^2 \pm q_2^2 \pm q_3^2 \) with \( q_i \in \mathbb{R}[x, y, z] \), and the 2-torsion points in the non-identity component of \( J(\mathbb{R}) \) correspond to the representations with + signs. This is the essential new result of the paper. For the proof one needs the following exact sequences

\[
0 \to \text{Pic}(Q_r) \to \text{Pic}(Q) \to \text{Br}(\mathbb{R}) \to \text{Br}(Q_r),
\]

\[
0 \geq J(\mathbb{R})^0 \to J(\mathbb{R})^0 \to \text{Br}(\mathbb{R}) \to 0,
\]

where \( Q_r \) is the curve \( Q \) as a curve over \( \mathbb{R} \) such that \( Q = Q_r \otimes \mathbb{C} \).

The first exact sequence follows from Hochschild-Serre spectral sequence for étale cohomology, the second sequence follows from a theorem of G. Weichold (1883; JFM 15.0431.01) reproved by Geyer (1964).

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MSC:
11E25 Sums of squares and representations by other particular quadratic forms
11E76 Forms of degree higher than two
14H40 Jacobians, Prym varieties

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