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On certain congruences for Fourier coefficients of classical cusp forms. (English)

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Connections between modular functions and hypergeometric series have been treated intermittently in the literature since the appearance of the classic work of Fricke and Klein on automorphic functions, the work of Poincaré early in the 20th century and the books of *C. Jordan* ["Cours d'analyze" (1959; Zbl 0092.27803)] and *L. R. Ford* ["Automorphic functions" (1929; JFM 55.0810.04)]. More recent instances are P. F. Stiller's work since the 1980's and two paper of *J. Hawkins* and the reviewer [Ill. J. Math. 36, 178–207 (1992; Zbl 0780.11027); Contemp. Math. 143, 451–475 (1993; Zbl 0801.11026)]. The article under review affords an interesting addition to the list (relatively short, considering the level of activity since Fricke-Klein in modular, automorphic functions and hypergeometric functions separately).

The author begins with the following observation, simple yet essential to the formulation of the paper's main result: Let $k \in 2\mathbb{Z}$, $k \geq 12$. Then any cusp form on $SL(2,\mathbb{Z})$ of weight k can be expressed uniquely in the form

$$f(z) = \Delta(z)^m E_4(z)^\delta E_6(z)^\varepsilon g\big(j(z)\big) = \sum_{n=1}^{\infty} \gamma(n) q^n,\ q = e^{2\pi i z}. \tag{1}$$

Here, Δ, E_4, E_6 and j are the familiar modular invariants, $k = 12m + 4\delta + 6\varepsilon$, with $m \in \mathbb{Z}^+$, $\delta = 0, 1$ or 2, $\varepsilon = 0$ or 1 and g is a polynomial over $\mathbb{C}$ of degree $\leq m-1$. (Note that any entire form on $SL(z,\mathbb{Z})$ can be written in the form (1) if one allows the degree of g to be $\leq m$.)

The ${}_2F_1$ hypergeometric series is defined by:

$${}_2F_1(a,b,c;x) = \sum_{m=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n, \tag{2}$$

where $(a)_n = a(a+1)\dots(a+n-1)$. With k, m, ε and g given by (1) (and so depending on the cusp form f), put

$$F(t) = t^m g(1/t)(1-1728t)^{(\varepsilon-1)/2} {}_2F_1\left(\frac{1}{12}, \frac{5}{12}, 1; 1728t\right)^{k-2} = \sum_{n=1}^{\infty} \alpha(n) t^n. \tag{3}$$

The principal result here expresses the congruence, modulo a prime p, between the $\gamma(p)$ of (1) and the $\alpha(p)$ of (3), provided that the coefficients of g lie in $\mathbb{Z}_p$:

Theorem. Let $g(t) \in \mathbb{C}[t]$ be associated to $f(z)$ by (1). If, furthermore, $g(t) \in \mathbb{Z}_p[t]$ (p a prime), then $\gamma(p) \equiv \alpha(p) \pmod{p}$. It follows from this that if the cusp form $f(z)$ is a normalized common eigenfunction of the Hecke operators, then for $r \in \mathbb{Z}^+$,

$$\gamma(p)\alpha(p^r) \equiv \alpha(p^{r+1}) + p^{k-1}\alpha(p^{r-1}) \pmod{p^{r+1}}. \tag{4}$$

The proof rests upon two lemmas. The first, a result due to *P. Stiller* [J. Number Theory 28, 219–232 (1988; Zbl 0644.10019), Theorems 3 and 4] relates the Eisenstein series $E_4(z)$ and $E_6(z)$ to special values of ${}_2F_1$, with $x = 1278/j(z)$; the second is a congruential result on formal power series with p-adic integral coefficients.

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33C05 Classical hypergeometric functions, ${}_2F_1$

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