Bonnet, Alexis; **David, Guy**
Cracktip is a global Mumford-Shah minimizer. (English) ⟮Zbl 1014.49009⟯
Astérisque. 274. Paris: Société Mathématique de France. vi, 259 p. (2001).

This remarkable monograph is devoted to the proof of a long-standing conjecture of De Giorgi, going back to 1991, about the global minimality for the Mumford-Shah image segmentation problem of the so-called "crack-tips". With a suitable choice of the reference frame in $\mathbb{R}^2$, a crack-tip is described in polar coordinates by $u(r,\theta) = \sqrt{2/\pi} r^{1/2} \sin(\theta/2)$, with $r > 0$ and $-\pi/2 < \theta < \pi/2$. In this case the discontinuity set K is the negative halfline and minimality is understood as follows:

$$\int_B |\nabla v|^2 dx + \mathcal{H}^1(C) \geq \int_B |\nabla u|^2 dx + \mathcal{H}^1(K \cap B)$$

for any ball B, any closed set $C \subset B$ and any $v \in C^1(B \setminus C)$ satisfying

$$\{u \neq v\} \cup (C \Delta K) \subset\subset B.$$

The proof is achieved by a technical tour de force, in which a suitable modification of the Mumford-Shah energy functional must be introduced and all the ideas used in previous papers by Bonnet, David, Dal Maso, Morel, Solimini come into play and are adapted to the modified energy functional. Assuming the non-minimality of the crack-tip, a better and minimizing competitor (v, C) is found and a contradiction is achieved by a careful analysis of the level sets of the conjugate harmonic function of v. This result might have a significant impact on the regularity theory for the Mumford-Shah functional, due to the obvious connection between classification of global minimizers and regularity. It is also worth to mention that the knowledge of the existence of a minimizer with a locally connected jump set, having a crack-tip in the interior of the domain, would lead to another proof of the conjecture on the lines of Bonnet's blow-up argument. Conversely, the existence of such minimizers directly follows by the minimality of the crack-tip.

Reviewer: Luigi Ambrosio (Pisa)

MSC:

49J45	Methods involving semicontinuity and convergence; relaxation	Cited in **14** Documents
49Q20	Variational problems in a geometric measure-theoretic setting	
49N60	Regularity of solutions in optimal control	
49K10	Optimality conditions for free problems in two or more independent variables	
49-02	Research exposition (monographs, survey articles) pertaining to calculus of variations and optimal control	

Keywords:

Mumford-Shah problem; global minimizers; regularity theory; image segmentation; crack-tip; energy functional