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Introduction to Λ-trees. (English) Zbl 1004.20014

Singapore: World Scientific. x, 315 p. (2001).

A tree (that is, a connected circuit-free graph) naturally defines an integer-valued path metric on its set of vertices. Some properties of this $\mathbb{Z}$-metric lead to the notion Λ-tree, where Λ is an arbitrary ordered Abelian group. In a tree, for any two segments with a common endpoint, their intersection is a segment, and, if the intersection is a singleton, their union is also a segment. A geodesic Λ-metric space with this property is called a Λ-tree. Here a Λ-metric space (X,d) is called geodesic if for every $x,y\in X$ there is a segment in X with endpoints x,y. A segment in X is defined to be the image under an isometry from a segment in Λ to X. So $\mathbb{Z}$-trees are just trees.

The theory of Λ-trees has its origin in the work of *R. C. Lyndon* on length functions in groups [Math. Scand. 12, 209-234 (1964; Zbl 0119.26402)]. The prototype such length function is given by taking a free group F with basis X, and defining $L(g)$ to be the length of the reduced word in $X^{\pm 1}$ representing an element g of F; then $L\colon F\to\mathbb{Z}$ is an integer-valued length function on F. For an ordered Abelian group Λ, a Λ-valued Lyndon length function on a group G is defined to be a function $L\colon G\to\Lambda$ satisfying certain natural properties of the standard length function on a free group. A free group acts on its Cayley graph with respect to a basis by left translations giving an action of the group by isometries on the corresponding $\mathbb{Z}$-tree. There is an equivalence between actions of a group on Λ-trees and Λ-valued length functions on the group. If G acts by isometries on a Λ-tree (X,d), and $x\in X$, then $L_x(g)=d(x,gx)$ gives a Λ-valued length function on G. If $L\colon G\to\Lambda$ is a length function on a group G satisfying some additional property (which holds for the length functions L_x above) then there are a Λ-tree (X,d), an element $x\in X$, and an action of G on X by isometries such that $L=L_x$.

Real-valued length functions were considered by *N. Harrison* [Trans. Am. Math. Soc. 174(1972), 77-106 (1973; Zbl 0255.20021)], and a construction of an $\mathbb{R}$-tree starting from length functions appeared in the author's paper [in Math. Proc. Camb. Philos. Soc. 80, 451-463 (1976; Zbl 0351.20024)]. *J. Tits*' paper [Contrib. to Algebra, Collect. Pap. dedic. E. Kolchin, 377-388 (1977; Zbl 0373.20039)] contains the first definition of an $\mathbb{R}$-tree. The importance of Λ-trees was established by *J. W. Morgan* and *P. B. Shalen* [Lect. Notes Math. 1167, 228-240 (1985; Zbl 0592.57007)], who showed how to compactify a generalization of Teichmüller space for a finitely generated group using $\mathbb{R}$-trees, and defined and studied Λ-trees for an arbitrary ordered Abelian group Λ. The theory was further developed in an important paper by *R. C. Lyndon* and *H. Bass* [Ann. Math. Stud. 111, 265-378 (1987; Zbl 0978.20500)], which became a standard reference.

The earlier book [Trees, Springer (1980; Zbl 0548.20018)] by *J.-P. Serre* contained the Bass-Serre theory of group actions on $\mathbb{Z}$-trees. The theory showed that from an action of a group on a $\mathbb{Z}$-tree essential information on the group can be obtained. This stimulated study of similar questions for actions of groups on Λ-trees. A group is called Λ-free if it has a free action without inversions on a Λ-tree. (An action of G on X is called free if $gx=x$ iff $g=1$, for any $g\in G$ and $x\in X$. An element g of G is said to be an inversion if g^2 has a fixed point but g does not.) J. W. Morgan and P. B. Shalen proved that the fundamental group of a closed surface is $\mathbb{R}$-free, except for the non-orientable surfaces of genus 1, 2, and 3. E. Rips' remarkable theorem asserts that any finitely generated $\mathbb{R}$-free group is the free product of finitely many groups, each of which is either a free Abelian group or the fundamental group of a closed surface. If a group is Λ-free for some Λ, it is called tree-free. Tree-free groups naturally arise in the model theory of free groups because models of the universal theory of free groups turn out to be tree-free. This fact is important in the light of the recent results in the model theory of free groups inspired by Tarski's problems (O. Kharlampovich and A. Myasnikov, Z. Sela).

The book under review is the first monograph devoted to Λ-trees and includes many results which were earlier dispersed in various papers (although it is not claimed to be a comprehensive and completely up-to-date account of the subject). The book will be useful for mathematicians and research students in topology, algebra, and model theory.

The book is organized as follows. Chapter 1 contains preliminaries, where the following topics are covered:

ordered Abelian groups; Λ-valued metrics and some metric spaces important in the context; graphs and simplicial trees; valuations. Chapters 2 and 3 contain the basics of the theory of Λ-trees, based on the paper by Alperin and Bass. The rest of the book is about group actions on Λ-trees, and is less self-contained. Chapter 4 is a collection of disjoint topics. A simplified proof of the author's following result is given: if a finitely generated group G has a non-trivial action on a Λ-tree for some Λ then G has a non-trivial action on an ℝ-tree. For certain group-theoretic properties, the author considers the restrictions on the actions on a Λ-tree, which a group with one of these properties can have; mostly results of Z. Khan and D. L. Wilkens are presented. For a field with valuation F, a method of constructing an action of $\mathrm{GL}_n(F)$ on a Λ-tree (which is due to J.-P. Serre for $\Lambda = \mathbb{Z}$ and to J. W. Morgan and P. B. Shalen in the general case) is described. Results of M. Culler and J. W. Morgan on group actions on ℝ-trees are presented, and, in particular, the result on compactness of the space of projectivized non-zero hyperbolic length functions for actions of G on ℝ-trees, for any finitely generated group G. In Chapter 5 free group actions on Λ-trees are studied. A proof is given that in a tree-free group any two-generated non-Abelian subgroup is free. This result is due to N. Harrison for ℝ-free groups; the general case is due to M. Urbański and L. Q. Zamboni, and the author. Results of V. N. Remeslennikov, A. M. Gaglione and D. Spellman, and the author on models of the universal theory of free groups are discussed; it is shown that these models are tree-free groups. A proof of Morgan-Shalen's theorem that the fundamental groups of closed surfaces are ℝ-free, with three exceptions, is given. In Chapter 6 Rips' theorem on classification of finitely generated ℝ-free groups is proven; the proof given is that of D. Gaboriau, G. Levitt, F. Paulin.

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MSC:

20E08 Groups acting on trees
20-02 Research exposition (monographs, survey articles) pertaining to group theory
20F65 Geometric group theory
05C05 Trees
05C25 Graphs and abstract algebra (groups, rings, fields, etc.)
57M05 Fundamental group, presentations, free differential calculus
57M07 Topological methods in group theory
20E05 Free nonabelian groups
20E06 Free products of groups, free products with amalgamation, Higman-Neumann-Neumann extensions, and generalizations
03C60 Model-theoretic algebra

Keywords:

Λ-trees; group actions on trees; Lyndon length functions; free groups; tree-free groups; ordered Abelian groups; fundamental groups; free products; model theory; universal theories