Deninger, Christopher
Deligne periods of mixed motives, K-theory and the entropy of certain $\mathbb{Z}^n$-actions. (English)
Zbl 0913.11027
J. Am. Math. Soc. 10, No. 2, 259-281 (1997).

The underlying paper relates aspects of the theory of dynamical systems to mixed motives. The link is suggested by integrals of the form $\int \log|f|\frac{dz}{z}$ which occur in older work of Beilinson on regulators relating algebraic K-theory to what is called Deligne-Beilinson cohomology (of analytic varieties). For $f = P$ a Laurent polynomial, it turns out that the above integrals can be interpreted as the entropy $h(\mathcal{O}_Z)$ of a $\mathbb{Z}^n$-action on the Pontryagin dual of the space of global sections of the structure sheaf $\mathcal{O}_Z$ of an irreducible closed subscheme Z of the split n-torus $\mathbb{G}^n_{m,\mathbb{Z}}$. This entropy (more precisely, its exponential) can then be identified with what is called the Mahler measure $m(P)$ (more precisely, $M(P) = \exp m(P)$) of the polynomial. In formula: for any $P \in \Gamma(\mathbb{G}^n_{m,\mathbb{Z}}, \mathcal{O}) = \mathbb{Z}[\mathbb{Z}^n] = \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, $P \neq 0$, and $Z = \operatorname{Spec}(\mathbb{Z}[\mathbb{Z}^n]/(P))$, one has

$$h(\mathcal{O}_Z) = m(P) := \frac{1}{(2\pi i)^n}\int_{T^n} \log|P(z_1,\dots,z_n)|\frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n},$$

where $T^n = S^1 \times \dots \times S^1$ is the real n-torus. In Beilinson's world, the associated $(n+1)$-fold cup product $\log|P| \cup \log|z_1| \cup \dots \cup \log|z_n|$ is just the image $r_{\mathcal{D}}(\{P, t_1, \dots, t_n\})$ of the regulator map of the K-theory symbol $\{P, t_1, \dots, t_n\}$ in Deligne-Beilinson cohomology of $\mathbb{G}^n_{m,\mathbb{Z}} \setminus Z$. It is known that, according to ideas of Deligne, algebraic K-theory should be related to mixed motives. In this respect, one may associate a mixed motive to the symbol $\{P, t_1, \dots, t_n\}$ and it is shown that the Deligne period of this mixed motive is just $m(P)$.

Take a non-zero Laurent polynomial $P \in \mathbb{Q}[\mathbb{Z}^n]$ and let $X_P/\mathbb{Q}$ be the complement of the zero locus of P in $\mathbb{G}^n_{m,\mathbb{Q}}$. X_P may be considered as a closed subvariety of $\mathbb{G}^{n+1}_{m,\mathbb{Q}}$ via the embedding of the graph of P. Now on $\mathbb{G}^{n+1}_m$ one has the action of $\Gamma_{n+1} := \mu_2^{n+1} \rtimes \mathfrak{S}_{n+1}$. One writes ε for the character which is the product on μ_2^{n+1} and the sign character of $\mathfrak{S}_{n+1}$. For any subgroup $\Gamma \subset \Gamma_{n+1}$ one may consider

$$X = \coprod_{\gamma\in\Gamma} X_P^\gamma \to \mathbb{G}^{n+1}_m, \quad \text{resp.} \quad X = \bigcup_{\gamma\in\Gamma} X_P^\gamma \to \mathbb{G}^{n+1}_m,$$

the sum, resp. the union, of the Γ-translates of X_P. In both cases, X is an affine n-dimensional closed subvariety of $\mathbb{G}^{n+1}_m$. Writing $H^\bullet(*)(\varepsilon)$ for the ε-isotypical component of $H^\bullet(*)$, one constructs an extension

$$0 \to H^n(X, n+1)(\varepsilon) \to N \to \mathbb{Q}(0) \to 0$$

and its twisted dual

$$0 \to \mathbb{Q}(1) \to M = N^\vee(1) \to H_n(X, -n)(\varepsilon) \to 0.$$

Let $\omega_{\mathcal{H}} \in F^0N_{dR}$ be the canonical class corresponding to $1 \in \mathbb{Q}$ under $F^0N_{dR} \xrightarrow{\sim} F^0\mathbb{Q}(0)_{dR} = \mathbb{Q}$. For the duals one has $M_B^+ \simeq H_n^B(X, \mathbb{Q}(-n))^+(\varepsilon)$. One has the period pairing $\langle,\rangle : M_B^+ \times F^0N_{dR} \to \mathbb{R}$. Let $i : X_P \hookrightarrow X$ be the inclusion, and write $i_*[T^n](\varepsilon)$ for the ε-isotypical component of $i_*[T^n]$ in $H_n(X(\mathbb{C}), \mathbb{Q})$. Finally, let $c \in M_B^+ \simeq H_n^B(X, \mathbb{Q}(-n))^+(\varepsilon)$ correspond to the cycle $i_*[T^n](\varepsilon)$. One has the result: Let $P \in \mathbb{Q}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ be without zeroes on T^n. Then, under the period pairing, $\langle c, \omega_{\mathcal{H}}\rangle = m(P)$. If $P \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ one gets an interpretation of $\langle c, \omega_{\mathcal{H}}\rangle$ as the entropy of a natural $\mathbb{Z}^n$-action. In A. Huber's category $\mathcal{MM}$ of mixed motives one has the Chern character

$$\text{ch} : H^{n+1}_{\mathcal{M}}(X_P, \mathbb{Q}(n+1)) \to \operatorname{Ext}^1_{\mathcal{MM}}(\mathbb{Q}(0), H^n(X_P, n+1)) = \operatorname{Ext}^1_{\mathcal{MM}}(\mathbb{Q}(0), H^n(X, n+1)(\varepsilon))$$

and one shows that $\text{ch}(\{P, t_1, \dots, t_n\})$ gives the extension N.

One can also consider symbols and their relation to differences of Mahler measures of the form $m(P^*) -$

$m(P)$, where $P^*(t_1,\dots,t_{n-1}) = a_{i_0}(t_1,\dots,t_{n-1})$ if

$$P(t_1,\dots,t_n) = \sum_{i\geq 0} a_i(t_1,\dots,t_{n-1})t_n^i \in \mathbb{C}[t_1^{\pm 1},\dots,t_n^{\pm 1}].$$

It turns out that such differences can also be expressed as evaluations of n-fold cup products in Deligne cohomology against suitable homology classes. This can also be given a motivic description. A particularly interesting example is the case $P(t_1,t_2) = t_1^2 t_2 + t_1 t_2^2 + t_1 t_2 + t_1 + t_2$, thus $P^*(t_1) = P(t_1, 0) = t_1$. The projective completion of the related variety Z turns out to be the elliptic curve $E : y^2 + xy + y = x^3 + x^2$, which is isogenous to $X_0(15)$. One has $m(P^*) = 0$ and $m(P)$ becomes (up to a minus sign) the evaluation of $r_{\mathcal{D}}([t_1,t_2])$ on an element of $H_1(E/\mathbb{R},\mathbb{Q}(-1))$ where $[t_1,t_2] \in H^2_{\mathcal{M}}(E,\mathbb{Q}(2))$. This leads to an expression of $m(P)$ in terms of an Eisenstein-Kronecker-Lerch series. This is closely related to $L(E,2)$ modulo $\mathbb{Q}^\times$ according to the Beilinson conjectures.

Reviewer: W. W. J. Hulsbergen (Haarlem)

MSC:

11G40 L-functions of varieties over global fields; Birch-Swinnerton-Dyer conjecture
19E08 K-theory of schemes
28D20 Entropy and other invariants
19F27 Étale cohomology, higher regulators, zeta and L-functions (K-theoretic aspects)

Keywords:

dynamical systems; mixed motives; Beilinson conjectures; algebraic K-theory; entropy; Mahler measure; Deligne period; Deligne cohomology

Full Text: DOI

References:

[1] Lars V. Ahlfors, Complex analysis: An introduction of the theory of analytic functions of one complex variable, Second edition, McGraw-Hill Book Co., New York-Toronto-London, 1966. · Zbl 0395.30001

[2] A. A. Beĭlinson, Higher regulators and values of \?-functions, Current problems in mathematics, Vol. 24, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984, pp. 181 – 238 (Russian).

[3] A. A. Beĭlinson, Higher regulators of modular curves, Applications of algebraic \?-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983) Contemp. Math., vol. 55, Amer. Math. Soc., Providence, RI, 1986, pp. 1 – 34.· doi:10.1090/conm/055.1/862627

[4] B. J. Birch and W. Kuyk , Modular functions of one variable. IV, Lecture Notes in Mathematics, Vol. 476, Springer-Verlag, Berlin-New York, 1975. · Zbl 0315.14014

[5] S. Bloch and D. Grayson, \?\(_{2}\) and \?-functions of elliptic curves: computer calculations, Applications of algebraic \?-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983) Contemp. Math., vol. 55, Amer. Math. Soc., Providence, RI, 1986, pp. 79 – 88.· doi:10.1090/conm/055.1/862631

[6] David W. Boyd, Speculations concerning the range of Mahler's measure, Canad. Math. Bull. 24 (1981), no. 4, 453 – 469. · Zbl 0474.12005 · doi:10.4153/CMB-1981-069-5

[7] D.W. Boyd, Explicit formulas for Mahler's measure. Preprint 1996.

[8] Pierre Deligne, Théorie de Hodge. II, Inst. Hautes Études Sci. Publ. Math. 40 (1971), 5 – 57 (French). Pierre Deligne, Théorie de Hodge. III, Inst. Hautes Études Sci. Publ. Math. 44 (1974), 5 – 77 (French).

[9] P. Deligne, Valeurs de fonctions \? et périodes d'intégrales, Automorphic forms, representations and \?-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977) Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 313 – 346 (French). With an appendix by N. Koblitz and A. Ogus.

[10] P. Deligne, Le groupe fondamental de la droite projective moins trois points, Galois groups over \? (Berkeley, CA, 1987) Math. Sci. Res. Inst. Publ., vol. 16, Springer, New York, 1989, pp. 79 – 297 (French). · Zbl 0742.14022 · doi:10.1007/978-1-4613-9649-9$_3$

[11] Christopher Deninger, Higher regulators and Hecke \?-series of imaginary quadratic fields. I, Invent. Math. 96 (1989), no. 1, 1 – 69. · Zbl 0721.14004 · doi:10.1007/BF01393970

[12] C. Deninger, Extensions of motives associated to symmetric powers of elliptic curves and to Hecke characters of imaginary quadratic fields. Preprint 1995. · Zbl 0888.14002

[13] C. Deninger, On extensions of mixed motives. Preprint 1995. · Zbl 0890.14001

[14] Christopher Deninger and Kay Wingberg, On the Beĭlinson conjectures for elliptic curves with complex multiplication, Beĭlinson's conjectures on special values of \?-functions, Perspect. Math., vol. 4, Academic Press, Boston, MA, 1988, pp. 249 – 272. · Zbl 0721.14006

[15] A. Huber, Mixed motives and their realizations in derived categories. Springer LNM 1604, 1995. · Zbl 0938.14008

[16] M. Rapoport, N. Schappacher, and P. Schneider , Beilinson's conjectures on special values of \?-functions, Perspectives in Mathematics, vol. 4, Academic Press, Inc., Boston, MA, 1988. · Zbl 0635.00005

[17] Uwe Jannsen, Mixed motives and algebraic \?-theory, Lecture Notes in Mathematics, vol. 1400, Springer-Verlag, Berlin, 1990. With appendices by S. Bloch and C. Schoen. · Zbl 0691.14001

[18] Uwe Jannsen, Steven Kleiman, and Jean-Pierre Serre , Motives, Proceedings of Symposia in Pure Mathematics, vol. 55, American Mathematical Society, Providence, RI, 1994. Uwe Jannsen, Steven Kleiman, and Jean-Pierre Serre , Motives, Proceedings of Symposia in Pure Mathematics, vol. 55, American Mathematical Society, Providence, RI, 1994.

[19] G. Kings, Extensions of motives of modular forms. To appear in Math. Ann. · Zbl 0913.11025

[20] Douglas Lind, Klaus Schmidt, and Tom Ward, Mahler measure and entropy for commuting automorphisms of compact groups, Invent. Math. 101 (1990), no. 3, 593 – 629. · Zbl 0774.22002 · doi:10.1007/BF01231517

[21] James S. Milne, Étale cohomology, Princeton Mathematical Series, vol. 33, Princeton University Press, Princeton, N.J., 1980. · Zbl 0433.14012

[22] Patrice Philippon, Critères pour l'indépendance algébrique, Inst. Hautes Études Sci. Publ. Math. 64 (1986), 5 – 52 (French). · Zbl 0615.10044

[23] M. Rapoport, N. Schappacher, and P. Schneider , Beilinson's conjectures on special values of \?-functions, Perspectives in Mathematics, vol. 4, Academic Press, Inc., Boston, MA, 1988. · Zbl 0635.00005

[24] Gary Alan Ray, Relations between Mahler's measure and values of \?-series, Canad. J. Math. 39 (1987), no. 3, 694 – 732. · Zbl 0621.12005 · doi:10.4153/CJM-1987-034-0

[25] K. Schmidt, Dynamical systems of algebraic origin. Progress in Math. 128, Basel 1995. CMP 95:16 · Zbl 0833.28001

[26] Norbert Schappacher and Anthony J. Scholl, The boundary of the Eisenstein symbol, Math. Ann. 290 (1991), no. 2, 303 – 321. , https://doi.org/10.1007/BF01459247 Norbert Schappacher and Anthony J. Scholl, Erratum: "The boundary of the Eisenstein symbol", Math. Ann. 290 (1991), no. 4, 815. · Zbl 0729.11027 · doi:10.1007/BF01459273

[27] Peter Schneider, Introduction to the Beĭlinson conjectures, Beĭlinson's conjectures on special values of \?-functions, Perspect. Math., vol. 4, Academic Press, Boston, MA, 1988, pp. 1 – 35.

[28] Anthony J. Scholl, Remarks on special values of \?-functions, \?-functions and arithmetic (Durham, 1989) London Math. Soc. Lecture Note Ser., vol. 153, Cambridge Univ. Press, Cambridge, 1991, pp. 373 – 392. · Zbl 0817.14007 · doi:10.1017/CBO9780511526053.016

[29] A. J. Scholl, Height pairings and special values of \?-functions, Motives (Seattle, WA, 1991) Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 571 – 598. · Zbl 0817.14008