Gelfand, I. M.; **Kapranov, M. M.**; **Zelevinsky, A. V.**
Discriminants, resultants, and multidimensional determinants. (English) Zbl 0827.14036
Mathematics: Theory & Applications. Boston, MA: Birkhäuser. vii, 523 p. (1994).

The book constitutes a major part of the renaissance of elimination theory, which, in favour of more abstract ideas, had been eliminated from algebra and geometry for several decades. Collecting and extending the fundamental and highly original results of the authors, it presents a unique blend of classical mathematics and very recent developments in algebraic geometry, homological algebra, and combinatorial theory.

The first part of the book is the general theory of discriminants and resultants. Chapter I introduces discriminants. Roughly speaking, the discriminant Δ_X of a projective variety $X \subset \mathbb{P}(V^*)$ is the equation of the dual variety $X^\vee \subset \mathbb{P}(V)$, provided codim $X^\vee = 1$, and 1 otherwise. The dual variety consists of the hyperplanes tangent to X. The general structural results proved in this chapter are the biduality theorem $X^{\vee\vee} = X$ and the Katz dimension formula. Several examples, especially plane curves, are studied in detail. – In chapter II the authors introduce the algebraic approach to discriminants, the 'Cayley method'. In this approach the dual variety of a smooth X is described as the variety of vectors for which the Koszul complex $\mathcal{K}(f)$ (or its dual) associated with the jet bundle of $\mathcal{O}_X(1)$ and its section defined by f is not exact. If $\mathcal{K}(f) \bigotimes \mathcal{M}$ is stably twisted for some line bundle $\mathcal{M}$, then Δ_X can (in principle) be computed as the determinant of the complex $H^0(\mathcal{K}(f) \bigotimes \mathcal{M})$; in the general case the discriminantal complex must be replaced by a spectral sequence. This approach yields formulas for the codimension and the degree of $X^\vee$ in terms of Hilbert polynomials of exterior powers of the tangent or cotangent bundle. In the classical cases of the Sylvester and Bezout formulas the determinant of the complex is the determinant of a matrix, and can be effectively computed. – Associated varieties, discussed in chapter III, are needed for the definition of general resultants. The associated variety $\mathcal{Z}(X)$ of an irreducible subvariety $X \subset \mathbb{P}^{n-1}$ of dimension k and degree d is the set of all $n-k-1$-dimensional projective subspaces of $\mathbb{P}^{n-1}$ meeting X; $\mathcal{Z}(X)$ is a hypersurface of degree d in the Grassmann variety $G(n-k, n)$. Its defining equation, the Chow form, is the resultant R_X of X. The 'Cayley trick' interprets R_X as the discriminant of a variety $\widetilde{X}$ by introducing auxiliary variables, i.e. $\widetilde{X} = X \times \mathbb{P}^{n-1}$. In the classical situation, in which one considers k forms of degree d in k variables, X is the d-th Veronese embedding of $\mathbb{P}^{k-1}$. The definition of resultants can be extended to the mixed case for which one works simultaneously with several embeddings of a variety X into projective spaces. The 'Cayley method' is applied to resultants, and for example yields the Bezout formula for the resultant of two binary forms and the Sylvester formula for the resultant of three ternary forms. – Chapter IV is devoted to Chow varieties, their embeddings into projective space via Chow forms (the Chow-van der Waerden theorem), and their defining equations. The Chow varieties of 0-cycles in projective space, for which the Chow-van der Waerden theorem gives a manageable system of equations, are discussed in detail. For cycles of positive dimension there exists a differential-geometric approach going back to Cayley and developed by Green and Morrison.

The second part of the book is of a much more combinatorial nature. It is governed by the 'A-philosophy', i.e. the study of A-discriminants and A-resultants. Here A is a finite set of (Laurent-) monomials, and the object of interest is the vector space $\mathbb{C}^A$ of polynomials spanned by them. An important example is the set A of all monomials of a fixed degree d. Chapter V gives a short, but rigorous introduction to toric varieties, starting from the projective toric variety X_A defined by A and proceeding to the most general case. – Newton polytopes are the subject of chapter VI. After a discussion of the theorems of Koushnirenko (with proof) and Bernstein, Chow polytopes are introduced. Roughly speaking, the Chow polytope of an algebraic cycle is the 'Newton polytope' of its Chow form (which however is an element of the homogeneous coordinate ring of a Grassmannian). – Chapter VII discusses the main combinatorial notions for the study of A- discriminants and A-resultants, namely the coherent triangulations of the polytope spanned by A with vertices in A. The coherent triangulations correspond via their characteristic functions to the vertices of the secondary polytope $\Sigma(A)$. The chapter is rich of examples of secondary polytopes for 'interesting' configurations A. – As a direct generalization of the classical notion, the A-resultant for a set A of monomials in $k-1$ variables gives the locus of those tuples $(f_1, \ldots, f_k)$, $f_i \in \mathbb{C}^A$ that have a common zero. Chapter VIII, specializing the general theory of resultants to that of A-resultants, describes the A-resultant in terms of the toric variety X_A. A major result is that the Chow polytope of X and the secondary polytope $\Sigma(A)$ coincide. – Chapter IX treats the A- discriminant Δ_A which can be chosen to be a polynomial over $\mathbb{Z}$. The heart of the chapter is an explicit description of the 'discriminantal complex' whose determinant yields Δ_A in terms of the differential forms on X_A. This complex also has a combinatorial description that leads to an expression for the degree of Δ_A by the volumes of the polytope $\operatorname{conv}(A)$ and its faces. – In chapters X and XI the Newton polytope of the A- discriminant is investigated; in the words of the authors, it is the 'magic crystal' that brings light to the concept of discriminant. For this purpose an auxiliary object, the principal A-determinant E_A is introduced. It has a factorization into the A'-discriminants where A' is extended over the 'faces' of A, and its Newton polytope is $\Sigma(A)$. The proofs of these theorems essentially occupy chapter X. – Chapter XI introduces another auxiliary object, the regular A- determinant D_A. If X_A is smooth, then $\Delta_A = D_A$. In general D_A is a rational function that can be factorized into the principal A-determinants of the 'faces' of A. If X_A is just quasi-smooth, then D_A is a polynomial. The proof of this result is based on Newton numbers of semigroups and cones. In the smooth case the vertices of the Newton polytope of Δ_A correspond to the so- called D-equivalence classes of coherent triangulations of $\operatorname{conv}(A)$ with vertices in A. The last subsection discusses the relations to real algebraic geometry and Hilbert's sixteenth problem, including a proof of Viro's theorem.

The third part of the book is devoted to classical discriminants and resultants. Chapter XII treats polynomials in one variable. It gives an overview of the classical formulas (with proofs independent of the general theory if possible) and contains a combinatorial description of the Newton polytope of the classical discriminant and resultant. – Discriminants and resultants of forms in several variables are the subject of chapter XIII. Cayley's determinantal formula is derived as the determinant of the resultant complex, and furthermore formulas for the resultant as a determinant of a single matrix are obtained (in certain cases) from Weyman's complexes. The chapter ends with an extension of the theory to the multihomogeneous case. – Finally, chapter XIV develops the theory of hyperdeterminants starting from Cayley's idea that defines the hyperdeterminant of an $(k_1+1)\times\cdots\times(k_r+1)$ (generalized) matrix as a discriminant, namely as the discriminant of the variety $\mathbb{P}^{k_1}\times\cdots\times\mathbb{P}^{k_r}$ embedded into the projective space $\mathbb{P}(V^*)$, $V = K^{k_1+1}\times\cdots\times K^{k_r+1}$ via the Segre map. The matrices of which the hyperdeterminant is taken are the elements of V. (There are other definitions of hyperdeterminants.) The hyperdeterminant is different from 1 if and only if each k_i is at most the sum of the others. After a discussion of its elementary algebraic properties, the degree of the hyperdeterminant is computed from the discriminantal complex. While being a complicated expression in general, it takes a rather simple form in the case of the 'boundary format' in which each k_i equals the sum of others. Similarly as the classical determinant, the hyperdeterminant of boundary format can also be described as a resultant. The chapter ends with a discussion of certain low-dimensional cases in which the hyperdeterminant is explicit known, for example from an application of Schläfli's method.

The appendix treats determinants of complexes and spectral sequences and contains a reproduction of Cayley's classical paper that already contains the description of the discriminant as the determinant of a Koszul complex.

Reviewer: W.Bruns

MSC:

14M25	Toric varieties, Newton polyhedra, Okounkov bodies
14-02	Research exposition (monographs, survey articles) pertaining to algebraic geometry
14M12	Determinantal varieties
13D25	Complexes (MSC2000)
13-01	Introductory exposition (textbooks, tutorial papers, etc.) pertaining to commutative algebra
14-01	Introductory exposition (textbooks, tutorial papers, etc.) pertaining to algebraic geometry
13C40	Linkage, complete intersections and determinantal ideals
15A15	Determinants, permanents, traces, other special matrix functions
13F20	Polynomial rings and ideals; rings of integer-valued polynomials
52B20	Lattice polytopes in convex geometry (including relations with commutative algebra and algebraic geometry)

Keywords:

Cayley method; determinant; elimination theory; discriminants; resultants; Chow varieties; toric varieties;

Newton polytopes; real algebraic geometry; hyperdeterminants; discriminant as the determinant of a Koszul complex