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PVTSI$^{(m)}$**: a novel approach to computation of Hadamard finite parts of nonperiodic singular integrals.** (English) Zbl 07462048

Calcolo 59, No. 1, Paper No. 7, 36 p. (2022)

Summary: We consider the numerical computation of $I[f] = \text{⨎}_a^b f(x)\,dx$, the Hadamard finite part of the finite-range singular integral $\int_a^b f(x)\,dx$, $f(x) = g(x)/(x-t)^m$ with $a < t < b$ and $m \in \{1, 2, \dots\}$, assuming that (i) $g \in C^\infty(a,b)$ and (ii) $g(x)$ is allowed to have arbitrary integrable singularities at the endpoints $x = a$ and $x = b$. We first prove that $\text{⨎}_a^b f(x)\,dx$ is invariant under any legitimate variable transformation $x = \psi(\xi)$, $\psi : [\alpha, \beta] \to [a, b]$, hence there holds $\text{⨎}_\alpha^\beta F(\xi)\,d\xi = \text{⨎}_a^b f(x)\,dx$, where $F(\xi) = f(\psi(\xi))\,\psi'(\xi)$. Based on this result, we next choose $\psi(\xi)$ such that $\mathcal{F}(\xi)$, the $\mathcal{T}$-periodic extension of $F(\xi)$, $\mathcal{T} = \beta - \alpha$, is sufficiently smooth, and prove, with the help of some recent extension/generalization of the Euler-Maclaurin expansion, that we can apply to $\text{⨎}_\alpha^\beta F(\xi)\,d\xi$ the quadrature formulas derived for periodic singular integrals developed in an earlier work of the author: [ibid. 58, No. 2, Paper No. 22, 24 p. (2021; Zbl 1472.65031)]. We give a whole family of numerical quadrature formulas for $\text{⨎}_\alpha^\beta F(\xi)\,d\xi$ for each m, which we denote $\widehat{T}^{(s)}_{m,n}[\mathcal{F}]$. Letting $G(\xi) = (\xi - \tau)^m F(\xi)$, with $\tau \in (\alpha, \beta)$ determined from $t = \psi(\tau)$, and letting $h = \mathcal{T}/n$, for $m = 3$, for example, we have the three formulas

$$\widehat{T}^{(0)}_{3,n}[\mathcal{F}] = h \sum_{j=1}^{n-1} \mathcal{F}(\tau + jh) - \frac{\pi^2}{3} G'(\tau)\,h^{-1} + \frac{1}{6} G'''(\tau)\,h,$$

$$\widehat{T}^{(1)}_{3,n}[\mathcal{F}] = h \sum_{j=1}^{n} \mathcal{F}(\tau + jh - h/2) - \pi^2 G'(\tau)\,h^{-1},$$

$$\widehat{T}^{(2)}_{3,n}[\mathcal{F}] = 2h \sum_{j=1}^{n} \mathcal{F}(\tau + jh - h/2) - \frac{h}{2} \sum_{j=1}^{2n} \mathcal{F}(\tau + jh/2 - h/4).$$

We show that all of the formulas $\widehat{T}^{(s)}_{m,n}[\mathcal{F}]$ converge to $I[f]$ as $n \to \infty$; indeed, if $\psi(\xi)$ is chosen such that $\mathcal{F}^{(i)}(\alpha) = \mathcal{F}^{(i)}(\beta) = 0$, $i = 0, 1, \dots, q-1$, and $\mathcal{F}^{(q)}(\xi)$ is absolutely integrable in every closed interval not containing $\xi = \tau$, then

$$\widehat{T}^{(s)}_{m,n}[\mathcal{F}] - I[f] = O(n^{-q}) \quad \text{as } n \to \infty,$$

where q is a positive integer determined by the behavior of $g(x)$ at $x = a$ and $x = b$ and also by $\psi(\xi)$. As such, q can be increased arbitrarily (even to $q = \infty$, thus inducing spectral convergence) by choosing $\psi(\xi)$ suitably. We provide several numerical examples involving nonperiodic integrands and confirm our theoretical results.

MSC:

41A55 Approximate quadratures
41A60 Asymptotic approximations, asymptotic expansions (steepest descent, etc.)
45B05 Fredholm integral equations
45E05 Integral equations with kernels of Cauchy type
65B05 Extrapolation to the limit, deferred corrections
65B15 Euler-Maclaurin formula in numerical analysis
65D30 Numerical integration
65D32 Numerical quadrature and cubature formulas

Keywords:

Hadamard finite part; singular integrals; hypersingular integrals; supersingular integrals; Euler-Maclaurin expansions; asymptotic expansions; variable transformation; numerical quadrature; trapezoidal rule

Full Text: DOI

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