zbMATH Open
THE FIRST RESOURCE FOR MATHEMATICS

Gross, Leonard

The Yang-Mills heat equation with finite action in three dimensions. (English) Zbl 1482.35001

Memoirs of the American Mathematical Society 1349. Providence, RI: American Mathematical Society (AMS) (ISBN 978-1-4704-5053-3/pbk; 978-1-4704-7015-9/ebook). v, 111 p. (2022).

The question of existence and uniqueness of solutions to the Yang-Mills heat equation is of intrinsic interest, partly because it is a naturally occurring quasilinear diffusion equation and partly because of the way that gauge invariance intervenes in the very formulation of the Cauchy problem. But, as in [*N. Charalambous* and the author, Commun. Math. Phys. 317, No. 3, 727–785 (2013; Zbl 1279.58005); J. Math. Phys. 56, No. 7, 073505, 21 p. (2015; Zbl 1321.81041)], this work is ultimately aimed at the construction of gauge invariant functions of distributional initial data by a completion procedure sketched in the introduction to [loc. cit., Zbl 1279.58005], in an anticipated application to quantum field theory. In order to construct local observables for this application the author is interested in solutions over a bounded open subset of $\mathbb{R}^3$, as well as over all of $\mathbb{R}^3$. The question of boundary conditions therefore arises. In [loc. cit.], Dirichlet, Neumann and Marini boundary conditions are considered. The latter consists in setting the normal component of the curvature to zero on the boundary. These are the only boundary conditions commensurate with the intended applications to quantum field theory. As the author mentions, solutions satisfying Marini boundary conditions are derived from solutions satisfying Neumann boundary conditions in a future work. In this paper the author only considers Dirichlet and Neumann boundary conditions. He proves the existence and uniqueness of solutions to the Yang-Mills heat equation over $\mathbb{R}^3$ and over a bounded open convex set in $\mathbb{R}^3$. The initial data is taken to lie in the Sobolev space of order one half, which is the critical Sobolev index for this equation over a three dimensional manifold. The existence is proven by solving first an augmented, strictly parabolic equation and then gauge transforming the solution to a solution of the Yang-Mills heat equation itself. The gauge functions needed to carry out this procedure lie in the critical gauge group of Sobolev regularity three halves, which is a complete topological group in a natural metric but is not a Hilbert Lie group. The nature of this group must be understood in order to carry out the reconstruction procedure. Solutions to the Yang-Mills heat equation are shown to be strong solutions modulo these gauge functions. Energy inequalities and Neumann domination inequalities are used to establish needed initial behavior properties of solutions to the augmented equation. This paper is organized as follows: Chapter 1 is an introduction to the subject. In Chapter 2, the author states the results obtained in this paper. Chapter 3 deals with solutions for the augmented Yang-Mills heat equation. Chapter 4 is devoted to initial behavior of solutions to the augmented equation, and Chapter 5 to gauge groups. Chapter 6 deals with the conversion group. Chapter 7 is devoted to the proof of the main result.

Reviewer: Ahmed Lesfari (El Jadida)

MSC:

35-02 Research exposition (monographs, survey articles) pertaining to partial differential equations
35K58 Semilinear parabolic equations
35K65 Degenerate parabolic equations
70S15 Yang-Mills and other gauge theories in mechanics of particles and systems
35K51 Initial-boundary value problems for second-order parabolic systems
58J35 Heat and other parabolic equation methods for PDEs on manifolds
81T13 Yang-Mills and other gauge theories in quantum field theory

Keywords:

weakly parabolic; gauge groups; Gaffney-Friedrichs inequality; Neumann domination; Dirichlet and Neumann boundary conditions; critical Sobolev index; critical gauge group

Full Text: DOI arXiv

References:

[1] Arnaudon, Marc; Bauer, Robert O.; Thalmaier, Anton, A probabilistic approach to the Yang-Mills heat equation, J. Math.

Pures Appl. (9), 81, 2, 143-166 (2002) · Zbl 1042.58021 · doi:10.1016/S0021-7824(02)01254-0

[2] Charalambous, Nelia; Gross, Leonard, The Yang-Mills heat semigroup on three-manifolds with boundary, Comm. Math. Phys., 317, 3, 727-785 (2013) · Zbl 1279.58005 · doi:10.1007/s00220-012-1558-0

[3] N. Charalambous and L. Gross, Neumann domination for the Yang-Mills heat equation, J. Math. Phys. 56 (2015), no. 7, 073505, 21. MR3405967 · Zbl 1321.81041

[4] Charalambous, Nelia; Gross, Leonard, Initial behavior of solutions to the Yang-Mills heat equation, J. Math. Anal. Appl., 451, 2, 873-905 (2017) · Zbl 06790506 · doi:10.1016/j.jmaa.2017.02.027

[5] Conner, P. E., The Neumann's problem for differential forms on Riemannian manifolds, Mem. Amer. Math. Soc., No. 20, 56 pp. (1956) · Zbl 0070.31404

[6] Dell'Antonio, Gianfausto; Zwanziger, Daniel, Every gauge orbit passes inside the Gribov horizon, Comm. Math. Phys., 138, 2, 291-299 (1991) · Zbl 0726.53067

[7] DeTurck, Dennis M., Deforming metrics in the direction of their Ricci tensors, J. Differential Geom., 18, 1, 157-162 (1983) · Zbl 0517.53044

[8] Donaldson, S. K., Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles, Proc. London Math. Soc. (3), 50, 1, 1-26 (1985) · Zbl 0529.53018 · doi:10.1112/plms/s3-50.1.1

[9] P.M.N. Feehan, Global existence and convergence of solutions to gradient systems and applications to Yang-Mills gradient flow, (2016), 475 pages, http://arxiv.org/abs/1409.1525v4.

[10] Freed, Daniel S., The geometry of loop groups, J. Differential Geom., 28, 2, 223-276 (1988) · Zbl 0619.58003

[11] Friedrichs, K. O., Differential forms on Riemannian manifolds, Comm. Pure Appl. Math., 8, 551-590 (1955) · Zbl 0066.07504 · doi:10.1002/cpa.3160080408

[12] Fujiwara, Daisuke, Concrete characterization of the domains of fractional powers of some elliptic differential operators of the second order, Proc. Japan Acad., 43, 82-86 (1967) · Zbl 0154.16201

[13] Gastel, Andreas, Singularities of first kind in the harmonic map and Yang-Mills heat flows, Math. Z., 242, 1, 47-62 (2002) · Zbl 1060.53071 · doi:10.1007/s002090100306

[14] Gastel, Andreas, Nonuniqueness for the Yang-Mills heat flow, J. Differential Equations, 187, 2, 391-411 (2003) · Zbl 1036.58011 · doi:10.1016/S0022-0396(02)00091-8

[15] Ginibre, J.; Velo, G., Global existence of coupled Yang-Mills and scalar fields in $(2+1)$-dimensional space-time, Phys. Lett. B, 99, 5, 405-410 (1981)· doi:10.1016/0370-2693(81)90558-X

[16] Ginibre, J.; Velo, G., The Cauchy problem for coupled Yang-Mills and scalar fields in the temporal gauge, Comm. Math. Phys., 82, 1, 1-28 (1981/82) · Zbl 0486.35048

[17] L. Gross, Stability for the Yang-Mills heat equation, (2017), 100 pages, http://arxiv.org/abs/1711.00114.

[18] L. Gross, The configuration space for Yang-Mills fields, In preparation, (2018b), 91 pages.

[19] L. Gross, Localization of solutions to the Yang-Mills heat equation, In preparation, (2018c), 20 pages.

[20] Grotowski, Joseph F., Finite time blow-up for the Yang-Mills heat flow in higher dimensions, Math. Z., 237, 2, 321-333 (2001) · Zbl 1009.53049 · doi:10.1007/PL00004871

[21] Grotowski, Joseph F.; Shatah, Jalal, Geometric evolution equations in critical dimensions, Calc. Var. Partial Differential Equations, 30, 4, 499-512 (2007) · Zbl 1127.58003 · doi:10.1007/s00526-007-0100-2

[22] Hong, Min-Chun; Tian, Gang, Global existence of the m-equivariant Yang-Mills flow in four dimensional spaces, Comm. Anal. Geom., 12, 1-2, 183-211 (2004) · Zbl 1068.58006

[23] Marini, Antonella, Dirichlet and Neumann boundary value problems for Yang-Mills connections, Comm. Pure Appl. Math., 45, 8, 1015-1050 (1992) · Zbl 0771.58007 · doi:10.1002/cpa.3160450806

[24] Marini, Antonella, Elliptic boundary value problems for connections: a non-linear Hodge theory, Mat. Contemp., 2, 195-205 (1992) · Zbl 0875.53002

[25] Marini, Antonella, The generalized Neumann problem for Yang-Mills connections, Comm. Partial Differential Equations, 24, 3-4, 665-681 (1999) · Zbl 0926.58020 · doi:10.1080/03605309908821437

[26] Marini, Antonella, Regularity theory for the generalized Neumann problem for Yang-Mills connections-non-trivial examples in dimensions 3 and 4, Math. Ann., 317, 1, 173-193 (2000) · Zbl 1001.58009 · doi:10.1007/s002080050363

[27] Mitrea, Marius, Dirichlet integrals and Gaffney-Friedrichs inequalities in convex domains, Forum Math., 13, 4, 531-567 (2001) · Zbl 0980.35040 · doi:10.1515/form.2001.021

[28] Morrey, Charles B., Jr.; Eells, James, Jr., A variational method in the theory of harmonic integrals. I, Ann. of Math. (2), 63, 91-128 (1956) · Zbl 0070.09901 · doi:10.2307/1969992

[29] Morrey, Charles B., Jr., A variational method in the theory of harmonic integrals. II, Amer. J. Math., 78, 137-170 (1956) · Zbl 0070.31402 · doi:10.2307/2372488

[30] Naito, Hisashi, Finite time blowing-up for the Yang-Mills gradient flow in higher dimensions, Hokkaido Math. J., 23, 3, 451-464 (1994) · Zbl 0827.53022 · doi:10.14492/hokmj/1381413099

[31] Oh, Sung-Jin, Gauge choice for the Yang-Mills equations using the Yang-Mills heat flow and local well-posedness in H^1, J. Hyperbolic Differ. Equ., 11, 1, 1-108 (2014) · Zbl 1295.35328 · doi:10.1142/S0219891614500015

[32] Oh, Sung-Jin, Finite energy global well-posedness of the Yang-Mills equations on $\mathbb{R}^{1+3}$: an approach using the Yang-Mills heat flow, Duke Math. J., 164, 9, 1669-1732 (2015) · Zbl 1325.35180 · doi:10.1215/00127094-3119953

[33] Pulemotov, Artem, The Li-Yau-Hamilton estimate and the Yang-Mills heat equation on manifolds with boundary, J. Funct. Anal., 255, 10, 2933-2965 (2008) · Zbl 1156.58010 · doi:10.1016/j.jfa.2008.07.025

[34] Råde, Johan, On the Yang-Mills heat equation in two and three dimensions, J. Reine Angew. Math., 431, 123-163 (1992) · Zbl 0760.58041 · doi:10.1515/crll.1992.431.123

[35] Ray, D. B.; Singer, I. M., \(R\)-torsion and the Laplacian on Riemannian manifolds, Advances in Math., 7, 145-210 (1971) · Zbl 0239.58014 · doi:10.1016/0001-8708(71)90045-4

[36] Renardy, Michael; Rogers, Robert C., An introduction to partial differential equations, Texts in Applied Mathematics 13, xiv+434 pp. (2004), Springer-Verlag, New York · Zbl 1072.35001

[37] L. A. Sadun, Continuum regularized Yang-Mills theory, Ph. D. Thesis, Univ. of California, Berkeley (1987), 67+ pages.

[38] Schlatter, Andreas E.; Struwe, Michael; Tahvildar-Zadeh, A. Shadi, Global existence of the equivariant Yang-Mills heat flow in four space dimensions, Amer. J. Math., 120, 1, 117-128 (1998) · Zbl 0938.58007

[39] Segal, I. E., The phase space for the Yang-Mills equations. Differential geometric methods in mathematical physics, Proc. Internat. Conf., Tech. Univ. Clausthal, Clausthal-Zellerfeld, 1978, Lecture Notes in Phys. 139, 101-109 (1981), Springer, Berlin-New York

[40] Struwe, Michael, Geometric evolution problems. Nonlinear partial differential equations in differential geometry, Park City, UT, 1992, IAS/Park City Math. Ser. 2, 257-339 (1996), Amer. Math. Soc., Providence, RI

[41] Taylor, Michael E., Partial differential equations. III, Applied Mathematical Sciences 117, xxii+608 pp. (1997), Springer-Verlag, New York

[42] Uhlenbeck, Karen K., Connections with \(L^p\) bounds on curvature, Comm. Math. Phys., 83, 1, 31-42 (1982) · Zbl 0499.58019

[43] Uhlenbeck, Karen K., The Chern classes of Sobolev connections, Comm. Math. Phys., 101, 4, 449-457 (1985) · Zbl 0586.53018

[44] Waldron, Alex, Instantons and singularities in the Yang-Mills flow, Calc. Var. Partial Differential Equations, 55, 5, Art. 113, 31 pp. (2016) · Zbl 1352.53058 · doi:10.1007/s00526-016-1062-z

[45] Weinkove, Ben, Singularity formation in the Yang-Mills flow, Calc. Var. Partial Differential Equations, 19, 2, 211-220 (2004) · Zbl 1060.53072 · doi:10.1007/s00526-003-0217-x

[46] Zwanziger, Daniel, Covariant quantization of gauge fields without Gribov ambiguity, Nuclear Phys. B, 192, 1, 259-269 (1981)· doi:10.1016/0550-3213(81)90202-9