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Least energy solutions to a cooperative system of Schrödinger equations with prescribed L^2-bounds: at least L^2-critical growth. (English) Zbl 07451515

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Summary: We look for least energy solutions to the cooperative systems of coupled Schrödinger equations

$$\begin{cases} -\Delta u_i + \lambda_i u_i = \partial_i G(u) & \text{in } \mathbb{R}^N,\ N \geq 3, \\ u_i \in H^1(\mathbb{R}^N), & i \in \{1,\dots,K\} \\ \int_{\mathbb{R}^N} |u_i|^2\,dx \leq \rho_i^2 \end{cases}$$

with $G \geq 0$, where $\rho_i > 0$ is prescribed and $(\lambda_i, u_i) \in \mathbb{R} \times H^1(\mathbb{R}^N)$ is to be determined, $i \in \{1,\dots,K\}$. Our approach is based on the minimization of the energy functional over a linear combination of the Nehari and Pohožaev constraints intersected with the product of the closed balls in $L^2(\mathbb{R}^N)$ of radii ρ_i, which allows to provide general growth assumptions about G and to know in advance the sign of the corresponding Lagrange multipliers. We assume that G has at least L^2-critical growth at 0 and admits Sobolev critical growth. The more assumptions we make about G, N, and K, the more can be said about the minimizers of the corresponding energy functional. In particular, if $K = 2$, $N \in \{3,4\}$, and G satisfies further assumptions, then $u = (u_1, u_2)$ is normalized, i.e., $\int_{\mathbb{R}^N} |u_i|^2\,dx = \rho_i^2$ for $i \in \{1,2\}$.

MSC:

35Q40 PDEs in connection with quantum mechanics
35Q55 NLS equations (nonlinear Schrödinger equations)
35Q60 PDEs in connection with optics and electromagnetic theory
35J20 Variational methods for second-order elliptic equations
78A60 Lasers, masers, optical bistability, nonlinear optics

Keywords:

Schrödinger equations; photonic crystal

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