Modin, Klas; Viviani, Milo

**Integrability of point-vortex dynamics via symplectic reduction: a survey.** (English)

Zbl 1482.37056
Arnold Math. J. 7, No. 3, 357-385 (2021).

The authors consider point-vertex dynamics and their integrability. This dynamics is described via idealized non-smooth solutions to the incompressible Euler equations on two-dimensional manifolds. The aim is to provide a unified treatment for proving integrability results for 2-, 3-, or 4-point-vertices. Part of their goal is to show how the symplectic reduction can provide a broader approach for proving integrability results, especially for point-vertex dynamics.

Euler equations on an orientable Riemannian manifold that govern an incompressible inviscid fluid have the form $\dot{\mathbf{v}} + \nabla_{\mathbf{v}}\mathbf{v} = -\nabla p$ with div $\mathbf{v} = 0$, where $\mathbf{v}$ is a vector field on a manifold $M$ incorporating the motion of the fluid's particles, $p$ is the pressure function and $\nabla_{\mathbf{v}}$ is the covariant derivative along $\mathbf{v}$. *H. Helmholtz* [J. Reine Angew. Math. 55, 25–55 (1858; ERAM 055.1448cj)] showed that the two-dimensional Euler equations have special solutions with a finite number of point-vertices. These solutions are not smooth and are characterized by the vorticity curl $\mathbf{v} = \sum_{i=1}^{n} \Gamma_i \delta_{\mathbf{r}_i}$ where non-zero $\Gamma_i$ is the strength of the vortex $i$, $\mathbf{r}_i$ is its position, and $\delta_{\mathbf{r}_i}$ is a delta function.

The authors describe point-vertex equations and their Hamiltonian structures on the sphere, the plane, the hyperbolic plane, and the flat torus. Each of these cases has a different symmetry group, and the symplectic reduction is treated separately in each case to establish integrability.

The authors also briefly review nonintegrability results. They conclude with some observations about how their results pertain to long term predictions for the Euler equations. An appendix offers a kind of visual portrait of point-vortex solutions.

Reviewer: William J. Satzer Jr. (St. Paul)

**MSC:**

37J35 Completely integrable finite-dimensional Hamiltonian systems, integration methods, integrability tests
37J30 Obstructions to integrability for finite-dimensional Hamiltonian and Lagrangian systems (nonintegrability criteria)
37J39 Relations of finite-dimensional Hamiltonian and Lagrangian systems with topology, geometry and differential geometry (symplectic geometry, Poisson geometry, etc.)
53D20 Momentum maps; symplectic reduction
70H06 Completely integrable systems and methods of integration for problems in Hamiltonian and Lagrangian mechanics
76B47 Vortex flows for incompressible inviscid fluids



**Keywords:**

point-vortex dynamics; integrable systems; Euler equations; symplectic reduction

**Full Text:** DOI arXiv

**References:**

[1] Aref, H., Point vortex dynamics: a classical mathematics playground, J. Math. Phys., 48, 65401, 23 (2007) · Zbl 1144.81308
[2] Arnold, VI, Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits, Ann. Inst. Fourier (Grenoble), 16, 319-361 (1966) · Zbl 0148.45301
[3] Arnold, VI; Khesin, B., Topological Methods in Hydrodynamics (1998), New York: Springer, New York · Zbl 0902.76001
[4] Bagrets, A.; Bagrets, D., Nonintegrability of two problems in vortex dynamics, Chaos, 7, 368-375 (1997) · Zbl 0933.37025
[5] Bolsinov, AV; Borisov, AV; Mamaev, IS, Lie algebras in vortex dynamics and celestial mechanics - IV, Regul. Chaotic Dyn.,

4, 23-50 (1999) · Zbl 1076.76513

[6] Bolsinov, AV; Jovanović, B., Noncommutative integrability, moment map and geodesic flows, Ann. Glob. Anal. Geom, 23, 305-322 (2003) · Zbl 1022.37038

[7] Borisov, A.V., Kilin, A.A., Mamaev, I.S.: Reduction and chaotic behavior of point vortices on a plane and a sphere. Nelineinaya Dinamika 233-246 (2005) · Zbl 1144.37458

[8] Bouchet, F.; Venaille, A., Statistical mechanics of two-dimensional and geophysical flows, Phys. Rep., 515, 227-295 (2012)

[9] Dritschel, DG; Qi, W.; Marston, JB, On the late-time behavior of a bounded, inviscid two-dimensional flow, J. Fluid Mech., 783, 1-22 (2015) · Zbl 1382.76119

[10] Eckhardt, B., Integrable four vortex motion, Phys. Fluid, 31, 2796-2801 (1988) · Zbl 0656.76026

[11] Euler, L., Principes généraux de l'état d'équilibre d'un fluide, Académie Royale des Sciences et des Belles-Lettres de Berlin, MémoiresAcadémie Royale des Sciences et des Belles-Lettres de Berlin, Mémoires, 11, 217-273 (1757)

[12] Gröbli, W., Specielle Probleme über die Bewegung geradliniger paralleler Wirbelfäden (1877), Zürich: Druck von Zürcher und Furrer, Zürich · Zbl 09.0675.01

[13] Hairer, E.; Lubich, C.; Wanner, G., Geometric Numerical Integration (2006), Berlin: Springer, Berlin · Zbl 1094.65125

[14] Helmholtz, H., Über integrale der hydrodynamischen gleichungen, welcher der wirbelbewegungen entsprechen, J. Reine Angew. Math., 55, 25-55 (1858)

[15] Hwang, S.; Kim, S-C, Point vortices on hyperbolic sphere, J. Geom. Phys., 59, 475-488 (2009) · Zbl 1166.76011

[16] Hwang, S.; Kim, S-C, Relative equilibria of point vortices on the hyperbolic sphere, J. Math. Phys., 54, 063101 (2013) · Zbl 1366.76015

[17] Khesin, B.; Misiołek, G., Euler and Navier-Stokes equations on the hyperbolic plane, Proc. Natl. Acad. Sci. (USA), 109, 18324-18326 (2012)

[18] Khesin, B.; Wendt, R., The Geometry of Infinite-Dimensional Groups (2009), Berlin: Springer, Berlin · Zbl 1160.22001

[19] Kilin, A. A., Artemova, E. M.: Integrability and chaos in vortex lattice dynamics. Regul. Chaotic Dyn. 101-113 (2019) · Zbl 1458.76021

[20] Kimura, Y., Vortex motion on surfaces with constant curvature, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 455, 245-259 (1999) · Zbl 0966.53046

[21] Kirchhoff, G., Vorlesungen über mathematische physik: mechanik (1876), Leipzig: BG Teubner, Leipzig · Zbl 08.0542.01

[22] Koiller, J.; Carvalho, S., Non-integrability of the 4-vortex system: analytical proof, Commun. Math. Phys., 120, 643-652 (1989) · Zbl 0825.58013

[23] Kraichnan, RH, Inertial ranges in two-dimensional turbulence, Phys. Fluid, 10, 1417-1423 (1967)

[24] Mamode, M.: Fundamental solution of the Laplacian on flat tori and boundary value problems for the planar Poisson equation in rectangles. Bound. Value Probl. 221 (2014) · Zbl 1304.35238

[25] Marchioro, C.; Pulvirenti, M., Mathematical Theory of Incompressible Nonviscous Fluids (2012), Berlin: Springer, Berlin · Zbl 0789.76002

[26] Marsden, JE; Misiołek, G.; Ortega, J-P; Perlmutter, M.; Ratiu, TS, Hamiltonian Reduction by Stages (2007), Berlin: Springer, Berlin · Zbl 1129.37001

[27] Marsden, JE; Ratiu, TS, Introduction to Mechanics and Symmetry (1998), Berlin: Springer, Berlin

[28] Marsden, JE; Weinstein, A., Co-adjoint orbits, vortices, and Clebsch variables for incompressible fluids, Phys. D, 7, 305-323 (1983) · Zbl 0576.58008

[29] McLachlan, RI; Modin, K.; Verdier, O., Symplectic integrators for spin systems, Phys. Rev. E, 89, 061301 (2014) · Zbl 1333.65139

[30] McLachlan, RI; Modin, K.; Verdier, O., A minimal-variable symplectic integrator on spheres, Math. Comp., 86, 2325-2344 (2016) · Zbl 1364.37166

[31] Miller, J., Statistical mechanics of Euler equations in two dimensions, Phys. Rev. Lett., 65, 2137-2140 (1990) · Zbl 1050.82553

[32] Modin, K.: Geometric hydrodynamics: from Euler, to Poincaré, to Arnold, 13th Young Researchers Workshop on Geometry, Mechanics and Control: Three Mini-courses, vol. 48 of Textos de Matematica, pp. 69-92, Departamento de Matematica da Universidade de Coimbra, Portugal (2019)

[33] Modin, K.; Verdier, O., Integrability of nonholonomically coupled oscillators, Discr. Contin. Dyn. Syst., 34, 1121-1130 (2014) · Zbl 1368.70023

[34] Modin, K.; Viviani, M., A Casimir preserving scheme for long-time simulation of spherical ideal hydrodynamics, J. Fluid Mech., 884, A22 (2020) · Zbl 1460.76182

[35] Modin, K.; Viviani, M., Lie-Poisson numerical schemes for isospectral flows, Found. Comput. Math., 20, 889-921 (2020) · Zbl 1450.37073

[36] Montaldi, J.; Nava-Gaxiola, C., Point vortices on the hyperbolic plane, J. Math. Phys., 55, 102702 (2014) · Zbl 1366.37089

[37] Newton, PK, The N-Vortex Problem: Analytical Techniques (2013), Berlin: Springer, Berlin

[38] Onsager, L., Statistical hydrodynamics, Il Nuovo Cimento (1943-1954), 6, 6, 279-287 (1949)

[39] Poincaré, H., Théorie des Tourbillons (1893), Paris: Carré, Paris · Zbl 25.1896.04

[40] Robert, R.; Sommeria, J., Statistical equilibrium states for two-dimensional flows, J. Fluid Mech., 229, 291-310 (1991) · Zbl

0850.76025

[41] Sakajo, T., The motion of three point vortices on a sphere, Jpn. J. Ind. Appl. Math., 16, 321-347 (1999) · Zbl 1306.76012

[42] Sakajo, T.: Integrable four-vortex motion on sphere with zero moment of vorticity. Phys. Fluid 19 (2007) · Zbl 1146.76521

[43] Stremler, MA; Aref, H., Motion of three point vortices in a periodic parallelogram, J. Fluid Mech., 392, 101-128 (1999) · Zbl 0954.76008

[44] Viviani, M., A minimal-variable symplectic method for isospectral flows, BIT Numer. Math., 60, 741-758 (2020) · Zbl 1448.37107

[45] Weiss, JB; McWilliams, JC, Nonergodicity of point vortices, Phys. Fluid. A, 3, 835-844 (1991) · Zbl 0732.76019

[46] Ziglin, SL, Nonintegrability of the problem of the motion of four point vortices, Dokl. Akad. Nauk SSSR, 250, 1296-1300 (1980) · Zbl 0464.76021

[47] Ziglin, SL, Quasi-periodic motions of vortex systems, Phys. D, 4, 261-269 (1982)