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Theories for incompressible rods: a rigorous derivation via Γ-convergence. (English)

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Summary: We use variational convergence to derive a hierarchy of one-dimensional rod theories, starting out from three-dimensional models in nonlinear elasticity subject to local volume-preservation. The densities of the resulting Γ-limits are determined by minimization problems with a trace constraint that arises from the linearization of the determinant condition of incompressibility. While the proofs of the lower bounds rely on suitable constraint regularization, the upper bounds require a careful, explicit construction of locally volume-preserving recovery sequences. After decoupling the cross-section variables with the help of divergence-free extensions, we apply an inner perturbation argument to enforce the desired non-convex determinant constraint. To illustrate our findings, we discuss the special case of isotropic materials.

MSC:

35Qxx Partial differential equations of mathematical physics and other areas of application

Keywords:

dimension reduction; Γ-convergence; Euler-Lagrange equations; incompressibility; rods

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