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**Finite-time singularity formation for  $C^{1,\alpha}$  solutions to the incompressible Euler equations on  $\mathbb{R}^3$ .** (English) Zbl 07441733

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Summary: It has been known since work of Lichtenstein and Gunther in the 1920s that the 3D incompressible Euler equation is locally well-posed in the class of velocity fields with Hölder continuous gradient and suitable decay at infinity. It is shown here that these local solutions can develop singularities in finite time, even for some of the simplest three-dimensional flows.

**MSC:**

35Q31 Euler equations

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**Keywords:**

Euler equations; singularity formation; asymptotic stability

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**References:**

- [1] Abidi, Hammadi; Hmidi, Taoufik; Keraani, Sahbi, On the global well-posedness for the axisymmetric {E}uler equations, *Math. Ann. Mathematische Annalen*, 347, 15-41 (2010) · [Zbl 1398.35159](#) · [doi:10.1007/s00208-009-0425-6](#)
- [2] Bardos, K.; Titi, È. S., Euler equations for an ideal incompressible fluid, *Uspekhi Mat. Nauk. Uspekhi Matematicheskikh Nauk*, 62, 5-46 (2007) · [Zbl 1139.76010](#) · [doi:10.1070/RM2007v062n03ABEH004410](#)
- [3] Beale, J. T.; Kato, T.; Majda, A., Remarks on the breakdown of smooth solutions for the  $\{(3\}\)-{D}$  {E}uler equations, *Comm. Math. Phys.. Communications in Mathematical Physics*, 94, 61-66 (1984) · [Zbl 0573.76029](#) · [doi:10.1007/BF01212349](#)
- [4] Buckmaster, Tristan; Vicol, Vlad, Convex integration and phenomenologies in turbulence, *EMS Surv. Math. Sci.. EMS Surveys in Mathematical Sciences*, 6, 173-263 (2019) · [Zbl 1440.35231](#) · [doi:10.4171/emss/34](#)
- [5] Chae, Dongho, Nonexistence of self-similar singularities for the 3{D} incompressible {E}uler equations, *Comm. Math. Phys.. Communications in Mathematical Physics*, 273, 203-215 (2007) · [Zbl 1157.35079](#) · [doi:10.1007/s00220-007-0249-8](#)
- [6] Chae, Dongho; Shvydkoy, Roman, On formation of a locally self-similar collapse in the incompressible {E}uler equations, *Arch. Ration. Mech. Anal.. Archive for Rational Mechanics and Analysis*, 209, 999-1017 (2013) · [Zbl 1285.35070](#) · [doi:10.1007/s00205-013-0630-z](#)
- [7] Choi, Kyudong; Hou, Thomas Y.; Kiselev, Alexander; Luo, Guo; Sverak, Vladimir; Yao, Yao, On the finite-time blowup of a one-dimensional model for the three-dimensional axisymmetric {E}uler equations, *Comm. Pure Appl. Math.. Communications on Pure and Applied Mathematics*, 70, 2218-2243 (2017) · [Zbl 1377.35218](#) · [doi:10.1002/cpa.21697](#)
- [8] Constantin, Peter, On the {E}uler equations of incompressible fluids, *Bull. Amer. Math. Soc. (N.S.). American Mathematical Society. Bulletin. New Series*, 44, 603-621 (2007) · [Zbl 1132.76009](#) · [doi:10.1090/S0273-0979-07-01184-6](#)
- [9] Constantin, Peter; Lax, P. D.; Majda, A., A simple one-dimensional model for the three-dimensional vorticity equation, *Comm. Pure Appl. Math.. Communications on Pure and Applied Mathematics*, 38, 715-724 (1985) · [Zbl 0615.76029](#) · [doi:10.1002/cpa.3160380605](#)
- [10] Constantin, Peter; Sun, Weiran, Remarks on {O}ldroyd-{B} and related complex fluid models, *Commun. Math. Sci.. Communications in Mathematical Sciences*, 10, 33-73 (2012) · [Zbl 1291.35201](#) · [doi:10.4310/CMS.2012.v10.n1.a3](#)
- [11] Constantin, Peter, The {E}uler equations and nonlocal conservative {R}iccati equations, *Internat. Math. Res. Notices. International Mathematics Research Notices*, 455-465 (2000) · [Zbl 0970.76017](#) · [doi:10.1155/S1073792800000258](#)
- [12] Constantin, Peter; Fefferman, Charles; Majda, Andrew J., Geometric constraints on potentially singular solutions for the  $\{(3\}\)-{D}$  {E}uler equations, *Comm. Partial Differential Equations. Communications in Partial Differential Equations*, 21, 559-571 (1996) · [Zbl 0853.35091](#) · [doi:10.1080/03605309608821197](#)
- [13] Danchin, R., Axisymmetric incompressible flows with bounded vorticity, *Uspekhi Mat. Nauk. Uspekhi Matematicheskikh Nauk*, 62, 73-94 (2007) · [Zbl 1139.76011](#) · [doi:10.1070/RM2007v062n03ABEH004412](#)
- [14] De Lellis, Camillo; Székelyhidi, Jr., László, On turbulence and geometry: from {N}ash to {O}nsager, *Notices Amer. Math. Soc.. Notices of the American Mathematical Society*, 66, 677-685 (2019) · [Zbl 1436.76009](#) · [doi:10.1090/noti1868](#)
- [15] Deng, Jian; Hou, Thomas Y.; Yu, Xinwei, Geometric properties and nonblowup of 3{D} incompressible {E}uler flow, *Comm. Partial Differential Equations. Communications in Partial Differential Equations*, 30, 225-243 (2005) · [Zbl 1142.35549](#) · [doi:10.1081/PDE-200044488](#)

- [16] Denisov, Sergey A., Infinite superlinear growth of the gradient for the two-dimensional {E}uler equation, *Discrete Contin. Dyn. Syst.. Discrete and Continuous Dynamical Systems. Series A*, 23, 755-764 (2009) · Zbl 1156.76009 · doi:10.3934/dcds.2009.23.755
- [17] Denisov, Sergey A., Double exponential growth of the vorticity gradient for the two-dimensional {E}uler equation, *Proc. Amer. Math. Soc.. Proceedings of the American Mathematical Society*, 143, 1199-1210 (2015) · Zbl 1315.35150 · doi:10.1090/S0002-9939-2014-12286-6
- [18] Do, Tam, On vorticity gradient growth for the axisymmetric 3{D} {E}uler equations without swirl, *Arch. Ration. Mech. Anal.. Archive for Rational Mechanics and Analysis*, 234, 181-209 (2019) · Zbl 1444.76024 · doi:10.1007/s00205-019-01388-3
- [19] Elgindi, Tarek M.; Ghoul, Tej-Eddine; Masmoudi, Nader, On the stability of self-similar blow-up for  $\alpha$  solutions to the incompressible {E}uler equations on  $\mathbb{R}^3$  (2019)
- [20] Elgindi, Tarek M.; Ghoul, Tej-Eddine; Masmoudi, Nader, Stable self-similar blow-up for a family of nonlocal transport equations, *Anal. PDE. Analysis & PDE*, 14, 891-908 (2021) · Zbl 1472.35277 · doi:10.2140/apde.2021.14.891
- [21] Elgindi, Tarek M.; Jeong, In-Jee, Finite-time singularity formation for strong solutions to the Boussinesq system, *Ann. PDE. Annals of PDE. Journal Dedicated to the Analysis of Problems from Physical Sciences*, 6, 5-50 (2020) · Zbl 1462.35287 · doi:10.1007/s40818-020-00080-0
- [22] Elgindi, Tarek M.; Jeong, In-Jee, On singular vortex patches, {I}: Well-posedness issues (2019) · Zbl 1454.35265
- [23] Elgindi, Tarek M.; Jeong, In-Jee, On the effects of advection and vortex stretching, *Arch. Ration. Mech. Anal.. Archive for Rational Mechanics and Analysis*, 235, 1763-1817 (2020) · Zbl 1434.35091 · doi:10.1007/s00205-019-01455-9
- [24] Elgindi, Tarek M., Remarks on functions with bounded Laplacian (2016)
- [25] Elgindi, Tarek M.; Jeong, In-Jee, Finite-time singularity formation for strong solutions to the axisymmetric 3{D} {E}uler equations, *Ann. PDE. Annals of PDE. Journal Dedicated to the Analysis of Problems from Physical Sciences*, 5, 16-51 (2019) · Zbl 1436.35055 · doi:10.1007/s40818-019-0071-6
- [26] Elgindi, Tarek M.; Masmoudi, Nader,  $L^\infty$  ill-posedness for a class of equations arising in hydrodynamics, *Arch. Ration. Mech. Anal.. Archive for Rational Mechanics and Analysis*, 235, 1979-2025 (2020) · Zbl 07170065 · doi:10.1007/s00205-019-01457-7
- [27] Gibbon, J. D., The three-dimensional {E}uler equations: where do we stand?, *Phys. D. Physica D. Nonlinear Phenomena*, 237, 1894-1904 (2008) · Zbl 1143.76389 · doi:10.1016/j.physd.2007.10.014
- [28] Gibbon, J. D.; Moore, D. R.; Stuart, J. T., Exact, infinite energy, blow-up solutions of the three-dimensional {E}uler equations, *Nonlinearity. Nonlinearity*, 16, 1823-1831 (2003) · Zbl 1040.35069 · doi:10.1088/0951-7715/16/5/315
- [29] Grafakos, Loukas; e~Silva, Diogo~Oliveira; Pramanik, Malabika; Seeger, Andreas; Stovall, Betsy, Some problems in harmonic analysis (2017)
- [30] Gunther, N., On the motion of fluid in a moving container, *Izvestia Akad. Nauk USSR, Ser. Fiz.-Mat.*, 20, 1323-1348 (1927) · Zbl 53.0786.08
- [31] Hou, Thomas Y.; Lei, Zhen, On the stabilizing effect of convection in three-dimensional incompressible flows, *Comm. Pure Appl. Math.. Communications on Pure and Applied Mathematics*, 62, 501-564 (2009) · Zbl 1171.35095 · doi:10.1002/cpa.20254
- [32] Jia, H.; Stewart, S.; Sverak, V., On the DeGregorio modification of the Constantin-Lax-Majda model, *Arch. Ration. Mech. Anal.. Archive for Rational Mechanics and Analysis*, 231, 1269-1304 (2019) · Zbl 1408.35152 · doi:10.1007/s00205-018-1298-1
- [33] Kato, Tosio; Ponce, Gustavo, Commutator estimates and the {E}uler and Navier-Stokes equations, *Comm. Pure Appl. Math.. Communications on Pure and Applied Mathematics*, 41, 891-907 (1988) · Zbl 0671.35066 · doi:10.1002/cpa.3160410704
- [34] Kato, Tosio, Remarks on the {E}uler and Navier-Stokes equations in  $\mathbb{R}^2$ . *Nonlinear Functional Analysis and its Applications, Part 2, Proc. Sympos. Pure Math.*, 45, 1-7 (1986)
- [35] Kerr, R. M., Evidence for a singularity of the three-dimensional, incompressible {E}uler equations. *Topological Aspects of the Dynamics of Fluids and Plasmas (Santa Barbara, CA, 1991)*, NATO Adv. Sci. Inst. Ser. E: Appl. Sci., 218, 309-336 (1992) · Zbl 0800.76081 · doi:10.1007/978-94-017-3550-6\_18
- [36] Kerr, Robert M., Evidence for a singularity of the three-dimensional, incompressible {E}uler equations, *Phys. Fluids A. Physics of Fluids A. Fluid Dynamics*, 5, 1725-1746 (1993) · Zbl 0800.76083 · doi:10.1063/1.858849
- [37] Kiselev, Alexander, Small scales and singularity formation in fluid dynamics (2018) · Zbl 1448.35398
- [38] Kiselev, Alexander; Ryzhik, Lenya; Yao, Yao; Zlatoš, Andrej, Finite time singularity for the modified SQG patch equation, *Ann. of Math. (2). Annals of Mathematics. Second Series*, 184, 909-948 (2016) · Zbl 1360.35159 · doi:10.4007/annals.2016.184.3.7
- [39] Kiselev, Alexander; Šverák, Vladimír, Small scale creation for solutions of the incompressible two-dimensional {E}uler equation, *Ann. of Math. (2). Annals of Mathematics. Second Series*, 180, 1205-1220 (2014) · Zbl 1304.35521 · doi:10.4007/annals.2014.180.3.9
- [40] Larios, Adam; Petersen, Mark R.; Titi, Edriss S.; Wingate, Beth, A computational investigation of the finite-time blow-up of the 3{D} incompressible {E}uler equations based on the Voigt regularization, *Theor. Comput. Fluid Dyn.. Theoretical and Computational Fluid Dynamics*, 32, 23-34 (2018) · doi:10.1007/s00162-017-0434-0
- [41] Larios, Adam; Titi, Edriss S., Global regularity versus finite-time singularities: some paradigms on the effect of boundary conditions and certain perturbations. *Recent Progress in the Theory of the {E}uler and Navier-Stokes Equations*, London Math. Soc. Lecture Note Ser., 430, 96-125 (2016) · Zbl 1408.35166 · doi:10.1017/CBO9781316407103.007
- [42] Lei, Zhen; Liu, Jie; Ren, Xiao, On the Constantin-Lax-Majda model with convection, *Comm. Math. Phys.. Communications in Mathematical Physics*, 375, 765-783 (2020) · Zbl 1439.35402 · doi:10.1007/s00220-019-03584-4
- [43] Lichtenstein, Leon, Über einige Existenzprobleme der Hydrodynamik homogener, unzusammendrückbarer, reibungsloser Flüssigkeiten und die Helmholtz'schen Wirbelsätze, *Math. Z.. Mathematische Zeitschrift*, 23, 89-154 (1925) · Zbl

- [44] Luo, Guo; Hou, Thomas Y., Potentially singular solutions of the 3{D} axisymmetric {E}uler equations, Proc. Nat. Acad. Sci., 111, 12968-12973 (2014) · Zbl 1431.35115 · doi:10.1073/pnas.1405238111
- [45] Luo, Guo; Hou, Thomas Y., Toward the finite-time blowup of the 3{D} axisymmetric {E}uler equations: a numerical investigation, Multiscale Model. Simul.. Multiscale Modeling & Simulation. A SIAM Interdisciplinary Journal, 12, 1722-1776 (2014) · Zbl 1316.35235 · doi:10.1137/140966411
- [46] Majda, Andrew J.; Bertozzi, Andrea L., Vorticity and Incompressible Flow, Cambridge Texts Appl. Math., 27, xii+545 pp. (2002) · Zbl 0983.76001 · doi:10.1017/CBO9780511613203
- [47] Nadirashvili, N. S., Wandering solutions of the two-dimensional {E}uler equation, Funktsional. Anal. i Prilozhen.. Akademiya Nauk SSSR. Funktsional'nyĭ Analiz i ego Prilozheniya, 25, 70-71 (1991) · Zbl 0769.35048 · doi:10.1007/BF01085491
- [48] Necas, J.; Ruzicka, M.; Sverák, V., On {L}eray's self-similar solutions of the {N}avier-{S}tokes equations, Acta Math.. Acta Mathematica, 176, 283-294 (1996) · Zbl 0884.35115 · doi:10.1007/BF02551584
- [49] Saint Raymond, X., Remarks on axisymmetric solutions of the incompressible {E}uler system, Comm. Partial Differential Equations. Communications in Partial Differential Equations, 19, 321-334 (1994) · Zbl 0795.35063 · doi:10.1080/03605309408821018
- [50] Sarria, Alejandro; Saxton, Ralph, Blow-up of solutions to the generalized inviscid {P}roudman-{J}ohnson equation, J. Math. Fluid Mech.. Journal of Mathematical Fluid Mechanics, 15, 493-523 (2013) · Zbl 1277.35077 · doi:10.1007/s00021-012-0126-x
- [51] Serfati, Philippe, Régularité stratifiée et équation d' {E}uler {3\}D à temps grand, C. R. Acad. Sci. Paris Sér. I Math.. Comptes Rendus de l'Académie des Sciences. Série I. Mathématique, 318, 925-928 (1994) · Zbl 0805.76009
- [52] Shirota, Taira; Yanagisawa, Taku, Note on global existence for axially symmetric solutions of the {E}uler system, Proc. Japan Acad. Ser. A Math. Sci.. Japan Academy. Proceedings. Series A. Mathematical Sciences, 70, 299-304 (1994) · Zbl 0831.35141 · doi:10.3792/pjaa.70.299
- [53] Stuart, J. T., Nonlinear {E}uler partial differential equations: singularities in their solution. Applied Mathematics, Fluid Mechanics, Astrophysics, 81-95 (1988)
- [54] Tao, Terence, On the universality of the incompressible {E}uler equation on compact manifolds, {II}. {N}on-rigidity of {E}uler flows, Pure Appl. Funct. Anal.. Pure and Applied Functional Analysis, 5, 1425-1443 (2020) · Zbl 1470.35273
- [55] Tao, Terence, On the universality of the incompressible {E}uler equation on compact manifolds, Discrete Contin. Dyn. Syst.. Discrete and Continuous Dynamical Systems. Series A, 38, 1553-1565 (2018) · Zbl 1397.35193 · doi:10.3934/dcds.2018064
- [56] Tao, Terence, Finite time blowup for {L}agrangian modifications of the three-dimensional {E}uler equation, Ann. PDE. Annals of PDE. Journal Dedicated to the Analysis of Problems from Physical Sciences, 2, 9-79 (2016) · Zbl 1397.35181 · doi:10.1007/s40818-016-0019-z
- [57] Tsai, Tai-Peng, On {L}eray's self-similar solutions of the {N}avier-{S}tokes equations satisfying local energy estimates, Arch. Rational Mech. Anal.. Archive for Rational Mechanics and Analysis, 143, 29-51 (1998) · Zbl 0916.35084 · doi:10.1007/s002050050099
- [58] Ukhovskii, M. R.; Iudovich, V. I., Axially symmetric flows of ideal and viscous fluids filling the whole space, J. Appl. Math. Mech.. Journal of Applied Mathematics and Mechanics, 32, 52-61 (1968) · doi:10.1016/0021-8928(68)90147-0
- [59] Yudovich, V. I., On the loss of smoothness of the solutions of the {E}uler equations and the inherent instability of flows of an ideal fluid, Chaos. Chaos. An Interdisciplinary Journal of Nonlinear Science, 10, 705-719 (2000) · Zbl 0982.76014 · doi:10.1063/1.1287066
- [60] Zlatoš, Andrej, Exponential growth of the vorticity gradient for the {E}uler equation on the torus, Adv. Math.. Advances in Mathematics, 268, 396-403 (2015) · Zbl 1308.35194 · doi:10.1016/j.aim.2014.08.012

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