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Phase space analysis of the Hermite semigroup and applications to nonlinear global well-posedness. (English) Zbl 1476.35297

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Summary: We study the Hermite operator $H=-\Delta+|x|^2$ in $\mathbb{R}^d$ and its fractional powers H^β, $\beta>0$ in phase space. Namely, we represent functions f via the so-called short-time Fourier, alias Fourier-Wigner or Bargmann transform $V_g f(g$ being a fixed window function), and we measure their regularity and decay by means of mixed Lebesgue norms in phase space of $V_g f$, that is in terms of membership to modulation spaces $M^{p,q}$, $0<p,q\leq\infty$. We prove the complete range of fixed-time estimates for the semigroup e^{-tH^β} when acting on $M^{p,q}$, for every $0<p,q\leq\infty$, exhibiting the optimal global-in-time decay as well as phase-space smoothing. As an application, we establish global well-posedness for the nonlinear heat equation for H^β with power-type nonlinearity (focusing or defocusing), with small initial data in modulation spaces or in Wiener amalgam spaces. We show that such a global solution exhibits the same optimal decay e^{-ct} as the solution of the corresponding linear equation, where $c=d^\beta$ is the bottom of the spectrum of H^β. Global existence is in sharp contrast to what happens for the nonlinear focusing heat equation without potential, where blow-up in finite time always occurs for (even small) constant initial data (constant functions belong to $M^{\infty,1}$).

MSC:

35R11 Fractional partial differential equations
35K15 Initial value problems for second-order parabolic equations
35K58 Semilinear parabolic equations
35S05 Pseudodifferential operators as generalizations of partial differential operators
42B35 Function spaces arising in harmonic analysis
47D06 One-parameter semigroups and linear evolution equations

Keywords:

Hermite operator; heat semigroup; modulation spaces; nonlinear heat equation

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