

**Shang Hsu, Yang**

**A hierarchically enhanced recovery-based error estimator for bidimensional elastoplastic modelling by generalized finite element approach.** (English) [Zbl 1476.74145](#)

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**Summary:** This paper aims to contribute with a hierarchically enhanced recovery-based error estimator by using the Zienkiewicz-Zhu (ZZ) recovery procedure and a generalized finite element formulation. The proposed stress recovery consists in an approximation procedure that is enriched hierarchically by using partition of unity (PU) and an enrichment function. In other words, the approximation matrix used for stress field recovery employs the enriched finite element (FE) shape function. Furthermore, the edge-bubble hierarchically enriched strategy in the context of a generalized finite element approach is adopted to develop the enriched  $C^0$  element and approximation the matrix. The error estimator based on this hierarchically enhanced recovery procedure and the generalized finite element approach is employed to solve several elastic and elastoplastic hardening benchmark problems. Moreover, the effectivity index, refinement index, recovered stress field, global error and local error, both in energy norm, are evaluated in several applications. In addition, the sensitivity of the proposed recovery procedure to a severely distorted mesh is also analysed. The results obtained by the proposed procedure is compared to other well-established recovery procedures and finite element approaches. By comparing the results, the robustness in the numerical performance, the versatility in the computational implementation and the competitiveness of the proposed procedure are demonstrated in the applications.

**MSC:**

[74S05](#) Finite element methods applied to problems in solid mechanics

[74C05](#) Small-strain, rate-independent theories of plasticity (including rigid-plastic and elasto-plastic materials)

**Keywords:**

Zienkiewicz-Zhu stress recovery procedure; partition of unity technique; enrichment function; edge-bubble strategy

**Software:**

[FEAPpv](#); [XFEM](#)

**Full Text:** [DOI](#)

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