

**Kusch, Jonas; Schlachter, Louisa**

**Oscillation mitigation of hyperbolicity-preserving intrusive uncertainty quantification methods for systems of conservation laws.** (English) Zbl 07396394

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**Summary:** In this article we study intrusive uncertainty quantification schemes for systems of conservation laws with uncertainty. While intrusive methods inherit certain advantages such as adaptivity and an improved accuracy, they suffer from two key issues. First, intrusive methods tend to show oscillations, especially at shock structures and second, standard intrusive methods can lose hyperbolicity. The aim of this work is to tackle these challenges with the help of two different strategies. First, we combine filters with the multi-element approach for the hyperbolicity-preserving stochastic Galerkin (hSG) scheme. While the limiter used in the hSG scheme ensures hyperbolicity, the filter as well as the multi-element ansatz mitigate oscillations. Second, we derive a multi-element approach for the intrusive polynomial moment (IPM) method. Even though the IPM method is inherently hyperbolic, it suffers from oscillations while requiring the solution of an optimization problem in every spatial cell and every time step. The proposed multi-element IPM method leads to a decoupling of the optimization problem in every multi-element. Thus, we are able to significantly decrease computational costs while improving parallelizability. Both proposed strategies are extended to adaptivity, allowing to adapt the number of basis functions in each multi-element to the smoothness of the solution. We finally evaluate and compare both approaches on various numerical examples such as a NACA airfoil and a nozzle test case for the two-dimensional Euler equations. In our numerical experiments, we observe the mitigation of spurious artifacts. Furthermore, using the multi-element ansatz for IPM significantly reduces computational costs.

**MSC:**

- 65-XX Numerical analysis
- 35L60 First-order nonlinear hyperbolic equations
- 35Q31 Euler equations
- 35Q62 PDEs in connection with statistics
- 37L65 Special approximation methods (nonlinear Galerkin, etc.) for infinite-dimensional dissipative dynamical systems
- 65M08 Finite volume methods for initial value and initial-boundary value problems involving PDEs
- 65M60 Finite element, Rayleigh-Ritz and Galerkin methods for initial value and initial-boundary value problems involving PDEs

Cited in **2** Documents

**Keywords:**

uncertainty quantification; polynomial chaos; stochastic Galerkin; filter; hyperbolicity; multi-element

**Software:**

HLLE; UQCreator

**Full Text:** [DOI](#) [arXiv](#)

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