

**Qi, Shijie**

**Normalized solutions for the Kirchhoff equation on noncompact metric graphs.** (English)

Zbl 07396377

Nonlinearity 34, No. 10, 6963-7004 (2021)

This work studies the nonlocal equation

$$-\left(a + b \int_{\mathcal{G}} |u'|^2 dx\right) u'' + \lambda u = |u|^{p-2} u, \quad x \in \mathcal{G}$$

with the constraint

$$\int_{\mathcal{G}} |u|^2 dx = \mu,$$

where  $\mathcal{G}$  is a compact metric graph, arising in the study of stationary solution of the model Kirchhoff and Carrier for vibrations of an elastic string. The problem has a natural variational formulation, which is used in its study. Detailed background on the problem and on previous results is given in the Introduction. The paper demonstrates the rich structure and complexity of the problem. It first studies the cases in which the graph  $\mathcal{G}$  is either a whole line or a half-line, showing that, depending on the value of  $p$  and of  $\mu$ , the equation can have no solution, a unique solution, or two solutions. Further, existence, non-existence and uniqueness results are given for non-compact metric graphs  $\mathcal{G}$  under a general topological assumption, as well as for specific graphs composed of half lines and a terminal edge.

Reviewer: [Guy Katriel \(Haifa\)](#)

**MSC:**

- 05C10 Planar graphs; geometric and topological aspects of graph theory
- 47J30 Variational methods involving nonlinear operators
- 34A09 Implicit ordinary differential equations, differential-algebraic equations
- 35R02 PDEs on graphs and networks (ramified or polygonal spaces)
- 35L72 Second-order quasilinear hyperbolic equations
- 49J40 Variational inequalities
- 81Q35 Quantum mechanics on special spaces: manifolds, fractals, graphs, lattices

**Keywords:**

[normalized solution](#); [Kirchhoff equation](#); [metric graph](#)

**Full Text:** [DOI](#)

**References:**

- [1] D'Ancona, P.; Spagnolo, S., Global solvability for the degenerate Kirchhoff equation with real analytic data, *Invent. Math.*, 108, 247-262 (1992) · [Zbl 0785.35067](#) · [doi:10.1007/bf02100605](#)
- [2] Arosio, A.; Panizzi, S., On the well-posedness of the Kirchhoff string, *Trans. Am. Math. Soc.*, 348, 305-330 (1996) · [Zbl 0858.35083](#) · [doi:10.1090/s0002-9947-96-01532-2](#)
- [3] Adami, R.; Dovetta, S.; Serra, E.; Tilli, P., Dimensional crossover with a continuum of critical exponents for NLS on doubly periodic metric graphs, *Anal. PDE*, 12, 1597-1612 (2019) · [Zbl 1414.35202](#) · [doi:10.2140/apde.2019.12.1597](#)
- [4] Adami, R.; Serra, E.; Tilli, P., NLS ground states on graphs., *Calc. Var. Partial Differ. Equ.*, 54, 743-761 (2015) · [Zbl 1330.35484](#) · [doi:10.1007/s00526-014-0804-z](#)
- [5] Adami, R.; Serra, E.; Tilli, P., Threshold phenomena and existence results for NLS ground states on metric graphs, *J. Funct. Anal.*, 271, 201-223 (2016) · [Zbl 1338.35448](#) · [doi:10.1016/j.jfa.2016.04.004](#)
- [6] Adami, R.; Serra, E.; Tilli, P., Negative energy ground states for the  $L^2$ -critical NLSE on metric graphs, *Commun. Math. Phys.*, 352, 387-406 (2017) · [Zbl 1372.35319](#) · [doi:10.1007/s00220-016-2797-2](#)
- [7] Adami, R.; Serra, E.; Tilli, P., Multiple positive bound states for the subcritical NLS equation on metric graphs, *Calc. Var. Partial Differ. Equ.*, 58, 5 (2019) · [Zbl 1405.35236](#) · [doi:10.1007/s00526-018-1461-4](#)

- [8] Adami, R.; Cacciapuoti, C.; Finco, D.; Noja, D., Constrained energy minimization and orbital stability for the NLS equation on a star graph, *Ann. Inst. Henri Poincaré C. Anal. Non Linéaire*, 31, 1289-1310 (2014) · [Zbl 1304.81087](#) · [doi:10.1016/j.anihpc.2013.09.003](#)
- [9] Bartsch, T.; Soave, N., Multiple normalized solutions for a competing system of Schrödinger equations, *Calc. Var. Partial Differ. Equ.*, 58, 22 (2019) · [Zbl 1409.35076](#) · [doi:10.1007/s00526-018-1476-x](#)
- [10] Bartsch, T.; Soave, N., A natural constraint approach to normalized solutions of nonlinear Schrödinger equations and systems, *J. Funct. Anal.*, 272, 4998-5037 (2017) · [Zbl 06714264](#) · [doi:10.1016/j.jfa.2017.01.025](#)
- [11] Bartsch, T.; Jeanjean, L.; Soave, N., Normalized solutions for a system of coupled cubic Schrödinger equations on  $\backslash(\#\#\#\backslash)$ , *J. Math. Pures Appl.*, 106, 583-614 (2016) · [Zbl 1347.35107](#) · [doi:10.1016/j.matpur.2016.03.004](#)
- [12] Bernstein, S., Sur une classe d'équations fonctionnelles aux dérivées partielles, (in Russian with French summary), *Bull. Acad. Sci. URSS, Set. Math.*, 4, 17-26 (1940) · [Zbl 0026.01901](#)
- [13] Bartsch, T.; Zhong, X.; Zou, W., Normalized solutions for a coupled Schrödinger system, *Math. Ann.*, 380, 1713 (2020) · [Zbl 1479.35762](#) · [doi:10.1007/s00208-020-02000-w](#)
- [14] Berkolaiko, G.; Kuchment, P., *Introduction to Quantum Graphs*, vol 186 (2013), Providence, RI: American Mathematical Society, Providence, RI · [Zbl 1318.81005](#)
- [15] Borrelli, W.; Carlone, R.; Tentarelli, L., Nonlinear Dirac equation on graphs with localized nonlinearities: bound states and nonrelativistic limits, *SIAM J. Math. Anal.*, 51, 1046-1081 (2019) · [Zbl 1411.35263](#) · [doi:10.1137/18m1211714](#)
- [16] Cazenave, T., *Semilinear Schrödinger Equations*, vol 10 (2003), Providence, RI: American Mathematical Society, Providence, RI · [Zbl 1055.35003](#)
- [17] Cacciapuoti, C.; Dovetta, S.; Serra, E., Variational and stability properties of constant solutions to the NLS equation on compact metric graphs, *Milan J. Math.*, 86, 305-327 (2018) · [Zbl 1404.35470](#) · [doi:10.1007/s00032-018-0288-y](#)
- [18] Carrier, G. F., On the non-linear vibration problem of the elastic string, *Q. Appl. Math.*, 3, 157-165 (1945) · [Zbl 0063.00715](#) · [doi:10.1090/qam/12351](#)
- [19] Dovetta, S., Existence of infinitely many stationary solutions of the  $L^2$ -subcritical and critical NLSE on compact metric graphs, *J. Differ. Equ.*, 264, 4806-4821 (2018) · [Zbl 1391.35382](#) · [doi:10.1016/j.jde.2017.12.025](#)
- [20] Dovetta, S., Mass-constrained ground states of the stationary NLSE on periodic metric graphs, *Nonlinear Differ. Equ. Appl.*, 26, 30 (2019) · [Zbl 1428.35650](#) · [doi:10.1007/s00030-019-0576-4](#)
- [21] Dovetta, S.; Serra, E.; Tilli, P., Uniqueness and non-uniqueness of prescribed mass NLS ground states on metric graphs, *Adv. Math.*, 374 (2020) · [Zbl 07258205](#) · [doi:10.1016/j.aim.2020.107352](#)
- [22] Dovetta, S.; Serra, E.; Tilli, P., NLS ground states on metric trees: existence results and open questions, *J. Lond. Math. Soc.*, 102, 1223-1240 (2020) · [Zbl 1462.35424](#) · [doi:10.1112/jlms.12361](#)
- [23] Dovetta, S.; Tentarelli, L., Ground states of the  $L^2$ -critical NLS equation with localized nonlinearity on a tadpole graph, *Discrete and continuous models in the theory of networks. Oper. Theory Adv. Appl.*, vol 281, 113-125 (2020), Springer: Cham, Springer · [Zbl 1473.35503](#)
- [24] Dovetta, S.; Tentarelli, L.,  $L^2$ -critical NLS on noncompact metric graphs with localized nonlinearity: topological and metric features, *Calc. Var. Partial Differ. Equ.*, 58, 108 (2019) · [Zbl 1432.35212](#) · [doi:10.1007/s00526-019-1565-5](#)
- [25] Dolbeault, J.; Esteban, M. J.; Laptev, A.; Loss, M., One-dimensional Gagliardo-Nirenberg-Sobolev inequalities: remarks on duality and flows, *J. Lond. Math. Soc.*, 90, 525-550 (2014) · [Zbl 1320.26017](#) · [doi:10.1112/jlms/jdu040](#)
- [26] Frantzeskakis, D. J., Dark solitons in atomic Bose-Einstein condensates: from theory to experiments, *J. Phys. A: Math. Theor.*, 43 (2010) · [Zbl 1192.82033](#) · [doi:10.1088/1751-8113/43/21/213001](#)
- [27] Gross, E. P., Structure of a quantized vortex in boson systems, *Nuovo Cimento*, 20, 454-477 (1961) · [Zbl 0100.42403](#) · [doi:10.1007/bf02731494](#)
- [28] Gou, T.; Jeanjean, L., Multiple positive normalized solutions for nonlinear Schrödinger systems, *Nonlinearity*, 31, 2319-2345 (2018) · [Zbl 1396.35009](#) · [doi:10.1088/1361-6544/aab0bf](#)
- [29] He, X.; Zou, W., Ground states for nonlinear Kirchhoff equations with critical growth, *Ann. Mat. Pura Appl.*, 193, 473-500 (2014) · [Zbl 1300.35016](#) · [doi:10.1007/s10231-012-0286-6](#)
- [30] He, X.; Zou, W., Existence and concentration behavior of positive solutions for a Kirchhoff equation in  $\backslash(\#\#\#\backslash)$ , *J. Differ. Equ.*, 252, 1813-1834 (2012) · [Zbl 1235.35093](#) · [doi:10.1016/j.jde.2011.08.035](#)
- [31] Jeanjean, L.; Luo, T.; Wang, Z-Q, Multiple normalized solutions for quasi-linear Schrödinger equations, *J. Differ. Equ.*, 259, 3894-3928 (2015) · [Zbl 1377.35074](#) · [doi:10.1016/j.jde.2015.05.008](#)
- [32] Kuchment, P., Quantum graphs: I. Some basic structures, *Waves Random Media*, 14, 107-128 (2004) · [Zbl 1063.81058](#) · [doi:10.1088/0959-7174/14/1/014](#)
- [33] Kirchhoff, G.; Hensel, K., *Vorlesungen Über Mathematische Physik*, vol 1 (1883), Leipzig: Teubner, Leipzig
- [34] Li, G.; Ye, H., On the concentration phenomenon of  $L^2$ -subcritical constrained minimizers for a class of Kirchhoff equations with potentials, *J. Differ. Equ.*, 266, 7101-7123 (2019) · [Zbl 1423.35106](#) · [doi:10.1016/j.jde.2018.11.024](#)
- [35] Mugnolo, D.; Noja, D.; Seifert, C., Airy-type evolution equations on star graphs, *Anal. PDE*, 11, 1625-1652 (2018) · [Zbl 06881629](#) · [doi:10.2140/apde.2018.11.1625](#)
- [36] Lions, J. L., *On some questions in boundary value problems of mathematical physics*, vol 30, 284-346 (1978)
- [37] Nhan, N. H.; Ngoc, L. T P.; Long, N. T., On a nonlinear wave equation of Kirchhoff-Carrier type: linear approximation and asymptotic expansion of solution in a small parameter, *Math. Probl. Eng.*, 2018, 1-18 (2018) · [Zbl 1427.35157](#) · [doi:10.1155/2018/3626543](#)

- [38] Pankov, A., Nonlinear Schrödinger equations on periodic metric graphs, *Discrete Contin. Dyn. Syst.*, 38, 697-714 (2018) · [Zbl 1374.35386](#) · [doi:10.3934/dcds.2018030](#)
- [39] Pelinovsky, D.; Schneider, G., Bifurcations of standing localized waves on periodic graphs, *Ann. Henri Poincaré*, 18, 1185-1211 (2017) · [Zbl 1387.35556](#) · [doi:10.1007/s00023-016-0536-z](#)
- [40] Pitaevskii, L., Vortex lines in an imperfect Bose gas, *Sov. Phys. JETP*, 13, 451-454 (1961)
- [41] Pohozaev, S., On a class of quasilinear hyperbolic equations, *Math. Sbornik*, 96, 152-166 (1975)
- [42] Qi, S.; Zou, W., Exact number of positive normalized solutions for the Kirchhoff equation
- [43] Soave, N., Normalized ground states for the NLS equation with combined nonlinearities, *J. Differ. Equ.*, 269, 6941-6987 (2020) · [Zbl 1440.35312](#) · [doi:10.1016/j.jde.2020.05.016](#)
- [44] Soave, N., Normalized ground states for the NLS equation with combined nonlinearities: the Sobolev critical case, *J. Funct. Anal.*, 279 (2020) · [Zbl 1440.35311](#) · [doi:10.1016/j.jfa.2020.108610](#)
- [45] Serra, E.; Tentarelli, L., Bound states of the NLS equation on metric graphs with localized nonlinearities, *J. Differ. Equ.*, 260, 5627-5644 (2016) · [Zbl 1408.35206](#) · [doi:10.1016/j.jde.2015.12.030](#)
- [46] Ye, H., The sharp existence of constrained minimizers for a class of nonlinear Kirchhoff equations, *Math. Methods Appl. Sci.*, 38, 2663-2679 (2015) · [Zbl 1331.35134](#) · [doi:10.1002/mma.3247](#)
- [47] Ye, H., The existence of normalized solutions for  $L^2$ -critical constrained problems related to Kirchhoff equations, *Z. Angew. Math. Phys.*, 66, 1483-1497 (2015) · [Zbl 1322.35032](#) · [doi:10.1007/s00033-014-0474-x](#)

This reference list is based on information provided by the publisher or from digital mathematics libraries. Its items are heuristically matched to zbMATH identifiers and may contain data conversion errors. It attempts to reflect the references listed in the original paper as accurately as possible without claiming the completeness or perfect precision of the matching.