Bechensteen, Arne Henrik; **Blanc-Féraud, Laure**; **Aubert, Gilles**
A continuous relaxation of the constrained $\ell_2 - \ell_0$ problem. (English) Zbl 07363956
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Summary: We focus on the minimization of the least square loss function under a k-sparse constraint encoded by a ℓ_0 pseudo-norm. This is a non-convex, non-continuous and NP-hard problem. Recently, for the penalized form (sum of the least square loss function and a ℓ_0 penalty term), a relaxation has been introduced which has strong results in terms of minimizers. This relaxation is continuous and does not change the global minimizers, among other favorable properties. The question that has driven this paper is the following: can a continuous relaxation of the k-sparse *constraint* problem be developed following the same idea and same steps as for the *penalized*$\ell_2 - \ell_0$ problem? We calculate the convex envelope of the constrained problem when the observation matrix is orthogonal and propose a continuous non-smooth, non-convex relaxation of the k-sparse constraint functional. We give some equivalence of minimizers between the original and the relaxed problems. The subgradient is calculated as well as the proximal operator of the new regularization term, and we propose an algorithm that ensures convergence to a critical point of the k-sparse constraint problem. We apply the algorithm to the problem of single-molecule localization microscopy and compare the results with well-known sparse minimization schemes. The results of the proposed algorithm are as good as the state-of-the-art results for the penalized form, while fixing the constraint constant is usually more intuitive than fixing the penalty parameter.

MSC:

68-XX Computer science
94-XX Information and communication theory, circuits

Keywords:

inverse problems; ℓ_0 problem; sparse modeling; non-convex; non-smooth; relaxation

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